✍ 2 *If* $A \in \mathbb{R}^{m\times n}$ *what are the dimensions of* $A^\dagger$*?,* $A^\dagger A$*?,* $AA^\dagger$*?*

Solution: *The dimension of* $A^\dagger A$ *is* $n \times m$ *and so* $A^\dagger A$*? is of size* $n \times n$*. Similarly,* $AA^\dagger$ *is of size* $m \times m$*.* □

✍ 3 *Show that* $A^\dagger A$ *is an orthogonal projector. What are its range and null-space?*

Solution: *One way to do this is to use the rank-one expansion:* $A = \sum \sigma_i u_i v_i^T$*. Then* $A^\dagger = \sum \frac{1}{\sigma_i} v_i u_i^T$ *and therefore,*

$$A^\dagger A = \left[\sum_{i=1}^{r} \frac{1}{\sigma_i} v_i u_i^T\right] \times \left[\sum_{j=1}^{r} \sigma_j u_j v_j^T\right] = \sum_{j=1}^{r} v_j v_j^T$$

which is a projector. □

✍ 4 *Same question for* $AA^\dagger$ *..*

Solution: *In this case we have*

$$AA^{\dagger} = \left[\sum_{j=1}^{r} \sigma_j u_j v_j^T\right] \left[\sum_{i=1}^{r} \frac{1}{\sigma_i} v_i u_i^T\right] \times = \sum_{j=1}^{r} u_j u_j^T$$

which is an orthogonal projector. □

✍ 5 *Consider the matrix:*

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

- *Compute the singular value decomposition of* A

Solution: *The nonzero singular values of* A *are the square roots of the eigenvalues of*

$$AA^T = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$$

These eigenvalues are 5 ± 4 *and so* $\sigma_1 = 3, \sigma_2 = 1$.

The matrix $\boldsymbol{U}$ of the left singular vectors is the matrix

$$\boldsymbol{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

If $\boldsymbol{A} = \boldsymbol{U\Sigma V^T}$, then $\boldsymbol{U' * A} = \boldsymbol{\Sigma V^T}$. Therefore to get $\boldsymbol{V}$ we use the relation: $\boldsymbol{V^T} = \boldsymbol{\Sigma^{-1} * U' * A}$. We have

$$\boldsymbol{U' * A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 4 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \boldsymbol{V^T} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1/3 & 0 & 4/3 & -1/3 \\ 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow$$

□

- *Find the matrix $\boldsymbol{B}$ of rank 1 which is the closest to $\boldsymbol{A}$ in 2-norm sense.*

Solution: *This is obtained by setting $\boldsymbol{\sigma_2}$ to zero in the SVD - or - equivalently as $\boldsymbol{B} = \boldsymbol{\sigma_1 u_1 v_1^T}$. You will find*

$$\boldsymbol{B} = \begin{pmatrix} 1/2 & 0 & 2 & -1/2 \\ -1/2 & 0 & -2 & 1/2 \end{pmatrix}$$

□

✍6 *Show that r_ϵ equals the number sing. values that are* $>\epsilon$

Solution: *This result is based on the following easy-to-prove extension of the Young=Eckhart theorem:*

$$\min_{rank(B)\le k} \|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1}$$

which implies that if $\|A - B\|_2 < \sigma_{k+1}$ *then* $rank(B)$ *must be* $> k$ *- or equivalently:*

$$\|A - B\|_2 < \sigma_k \rightarrow rank(B) \ge k.$$

Let k *be the number that satisfies* $\sigma_{k+1} \le \epsilon < \sigma_k$ *– which is the number of sing. values that are* $> \epsilon$*. Then we see from the above that* $\|A - B\|_2 \le \epsilon$ *implies that* $rank(B) \ge k$*. The smallest possible rank for* B *is precisely the integer* k *defined above.* □