✍2 *Show that* $\bar{X} = X(I - \frac{1}{n}ee^T)$ *(here* e *= vector of all ones). What does the projector* $(I - \frac{1}{n}ee^T)$ *do?*

Solution: *Each column of* $\bar{X}$ *is* $\bar{x} = x - \mu$ *so that* $\bar{X} = X - \mu e^T$*, where* μ *is the sample mean. But we have* $\mu = \frac{1}{n}\sum x_i = \frac{1}{n}Xe$ *and so,*

$$\bar{X} = X - \frac{1}{n}Xee^T = X[I - \frac{1}{n}ee^T]$$

The matrix $(I - \frac{1}{n}ee^T)$ *represents a projector that centers the data so the mean is zero.*

✍3 *Show that solution* V *also minimizes 'reconstruction error' ..*

Solution: *The main property that is exploited in the proof is the fact that* $Tr(ABC) = Tr(BCA)$ *(when dimensions are compatible). First we note that* $\sum_i \|\bar{x}_i - VV^T\bar{x}_i\|^2 = \|(I - VV^T)X\_F^2$*. We will call* P *the pojector* $P = VV^T$*. Then:*

$$
\begin{aligned}
\|(I - VV^T)X\_F^2 * &= \mathrm{Tr}(I - P)XX^T(I - P) \\
&= \mathrm{Tr}(XX^T - PXX^T)(I - P) \\
&= \mathrm{Tr}(XX^T) - \mathrm{Tr}(PXX^T) - \mathrm{Tr}(XX^TP) + \mathrm{Tr}(PXX^TP) \\
&= \mathrm{Tr}(XX^T) - \mathrm{Tr}(PXX^T) - \mathrm{Tr}(XX^TP) + \mathrm{Tr}(XX^TP^2) \\
&= \mathrm{Tr}(XX^T) - \mathrm{Tr}(PXX^T) - \mathrm{Tr}(XX^TP) + \mathrm{Tr}(XX^TP) \\
&= \mathrm{Tr}(XX^T) - \mathrm{Tr}(PXX^T) \\
&= \mathrm{Tr}(XX^T) - \mathrm{Tr}(VV^TXX^T) \\
&= \mathrm{Tr}(XX^T) - \mathrm{Tr}(V^TXX^TV)
\end{aligned}
$$

The first term is a constant, therefore the minimum is reached when the maxiumn of the second term is reached. □

✍ 4 ... *and that it also maximizes* $\sum_{i,j} \|y_i - y_j\|_2^2$

Solution: *Let us denote by $\bar{y}$ the sample mean of the j_js, i.e.,*

$$\bar{y} = \frac{1}{n}\sum_{j=1}^{n} y_j.$$

We proceed backward examine the sum $\sum_{i,j} \|y_i - y_j\|_2^2$

$$\begin{aligned}
\sum_{i,j} \|y_i - y_j\|_2^2 &= \sum_{i,j} \|(y_i - \bar{y}) - (y_j - \bar{y})\|_2^2 \\
&= \sum_{i,j} ((y_i - \bar{y}) - (y_j - \bar{y}), (y_i - \bar{y}) - (y_j - \bar{y})) \\
&= \sum_i \sum_j \left[\|(y_i - \bar{y})\|_2^2 + \|(y_j - \bar{y})\|_2^2\right] \cdots \\
&\quad - 2\sum_i \sum_j ((y_i - \bar{y}), (y_j - \bar{y})) \\
&= 2n\sum_i \|y_i - \bar{y}\|_2^2 - 2\sum_i (y_i - \bar{y}, \sum_j (\bar{y} - y_j)) \\
&= 2n\sum_i \|y_i - \bar{y}\|_2^2
\end{aligned}$$

The last equality comes from the fact that $\sum_j (\bar{y} - y_j) = 0$ $\square$

Further reading: Some references on applications of the SVD

For image processing:

https://arxiv.org/pdf/1211.7102.pdf

The extraordinary SVD

https://arxiv.org/pdf/1103.2338.pdf%22%20rel=%22nofollow

An outstanding paper for understanding the SVD and a few of its applications:

https://sites.math.washington.edu/~morrow/464_16/svd.pdf

An old paper of ours that discusses a form of truncated SVD for face recognition:

https://www-users.cs.umn.edu/~saad/PDF/umsi-2006-16.pdf