✍ 1 *Consider*

$$A = \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

Eigenvalues of A? their algebraic multiplicities? their geometric multiplicities? Is one a semi-simple eigenvalue?

Solution: *The eigenvalues of A are 1, and 2. The algebraic multiplicity of 1 is 2. To get the geometric multiplicity of the eigenvalue $\lambda = 1$ we need to eigenvectors. For this we need to solve:*

$$\begin{pmatrix} 0 & 2 & -4 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} u = 0.$$

There is only one solution vector (up to a product by a scalar) namely:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So the geometric multiplicity is one. $\square$

✍ 2 *Same questions if a_{33} is replaced by one.*

Solution: *The matrix become*

$$A = \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

and now we have one eigenvalue algebraic multiplicity 3.

To get the geometric multiplicity of the eigenvalue $\lambda = 1$ we need to eigenvectors. For this we *need to solve:*

$$\begin{pmatrix} 0 & 2 & -4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} u = 0.$$

we still get a geometric mult. of 1. □

✍3 *Same questions if in addition a_{12} is replaced by zero.*

Solution: Solution: *The matrix become*

$$A = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

and we also have one eigenvalue with algebraic multiplicity 3. The geometric multiplicity increases to 2. □

✍4 *Show that there is at least one eigenvalue and eigenvector of A: $Ax = \lambda x$, with $\|x\|_2 = 1$*

Solution: *This comes from the fact that the equation $P_A(\lambda) = \det(A - \lambda I) = 0$ is a* *polynomial equation and as such it must have at least one root - a well-known result.* ☐

✍ 5 *There is a unitary transformation* $\boldsymbol{P}$ *such that* $\boldsymbol{Px} = \boldsymbol{e}_1$. *How do you define* $\boldsymbol{P}$?

Solution: *This is just the Householder transform.. See Lecture notes set number 8.* ☐

✍ 6 *Show that* $\boldsymbol{PAP}^H = \left(\begin{array}{c|c} \boldsymbol{\lambda} & ** \\ \hline \boldsymbol{0} & \boldsymbol{A}_2 \end{array}\right)$.

Solution: *This is equivalent to showing that* $\boldsymbol{PAP}^H\boldsymbol{e}_1 = \boldsymbol{\lambda e}_1$. *We have*

$$\boldsymbol{PAP}^H\boldsymbol{e}_1 = \boldsymbol{PAPe}_1 = \boldsymbol{P}(\boldsymbol{Ax}) = \boldsymbol{P}(\boldsymbol{\lambda x}) = \boldsymbol{\lambda Px} = \boldsymbol{\lambda e}_1$$

☐

✍ 9 *Another proof altogether: use Jordan form of* $\boldsymbol{A}$ *and QR factorization* **Solution:** *Jordan form:*

$$\boldsymbol{A} = \boldsymbol{XJX}^{-1}$$

Let $\boldsymbol{X} = \boldsymbol{Q}\boldsymbol{R}_0$ *then:*

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}_0\boldsymbol{J}\boldsymbol{R}_0^{-1}\boldsymbol{Q}^H \equiv \boldsymbol{Q}\boldsymbol{R}\boldsymbol{Q}^H \quad \textit{with} \quad \boldsymbol{R} = \boldsymbol{R}_0\boldsymbol{J}\boldsymbol{R}_0^{-1}$$

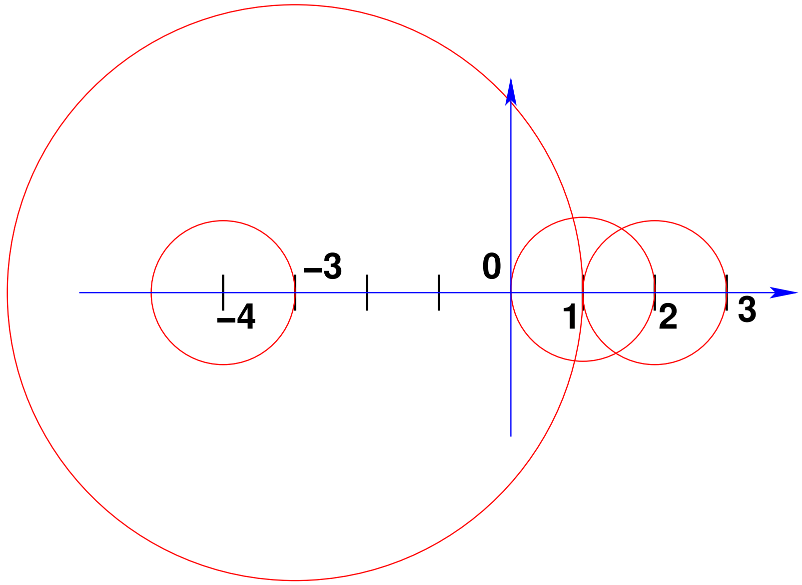
□

✍ 10 *Find a region of the complex plane where the eigenvalues of the following matrix are located:*

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & -2 & -3 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & -4 \end{pmatrix}$$

Solution: *Use Gershgorin's theorem. There are 4 disks:*

$$\begin{aligned} D_1 &= D(1,1); & D_2 &= D(2,1) \\ D_3 &= D(-3,4); & D_4 &= D(-4,1) \end{aligned}$$



The last disk is included in the 3rd. The spectrum is included in the union of the 3 other disks. ☐

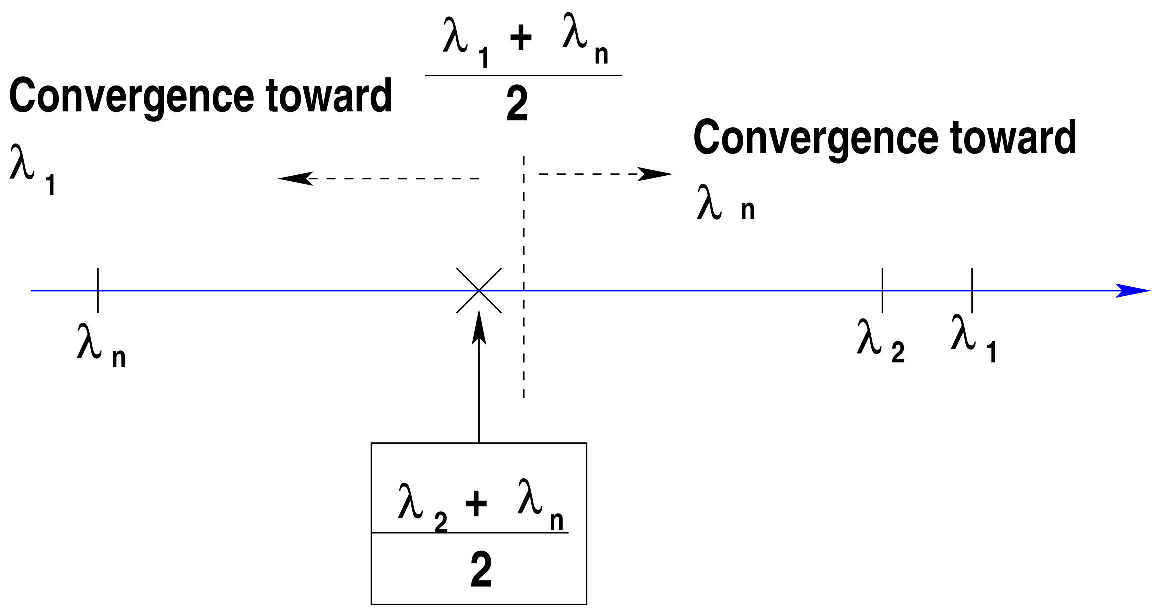
✍ 11 *Convergence factor $\phi(\sigma)$ as a function of σ.*

Solution: *The eigenvalues of the shifted matrix are $\lambda_i - \sigma$. When $\sigma > (\lambda_1 + \lambda_n)/2$ then the algorithm will converge toward λ_n because $|\lambda_n - \sigma| > |\lambda_1 - \sigma|$. We will ignore this case.*

Assume now that $\sigma < (\lambda_1 + \lambda_n)/2$. If $\sigma < (\lambda_2 + \lambda_n)/2$ then largest eigenvalue of $A - \sigma$ is $\lambda_1 - \sigma$ and second largest is $\lambda_2 - \sigma$. If $\sigma \geq (\lambda_2 + \lambda_n)/2$ then largest eigenvalue of $A - \sigma$

is $\lambda_n - \sigma$ *and second largest is* $\lambda_2 - \sigma$. *Therefore, setting* $\mu = (\lambda_2 + \lambda_n)/2$, *we get*

$$\phi(\sigma) = \begin{cases} \frac{|\lambda_2 - \sigma|}{|\lambda_1 - \sigma|} = \frac{\lambda_2 - \sigma}{\lambda_1 - \sigma} & \text{if } \sigma < \mu \\ \frac{|\lambda_n - \sigma|}{|\lambda_1 - \sigma|} = \frac{\sigma - \lambda_n}{\lambda_1 - \sigma} & \text{if } \sigma > \mu \end{cases}$$



Note that for $\sigma < \mu$ *we have* $\phi(\sigma) = 1 - (\lambda_1 - \lambda_2)/(\lambda_1 - \sigma)$ *which is a decreasing function while when* $\sigma > \mu$ *we have* $\phi(\sigma) = -1 + (\lambda_1 - \lambda_n)/(\lambda_1 - \sigma)$ *which is an increasing function. The min. is reached when these 2 values are equal which leads to the solution* $\sigma_{opt} = (\lambda_n + \lambda_2)/2$

□

Additional notes.

In discussing Gerschgorin theorem it was stated:

➤ *Refinement: if disks are all disjoint then each of them contains one eigenvalue*

Question: Why?

Solution:

Consider the matrix $A(t) = D + t(A - D)$ *where* D *is the diagonal of* A*. Note* $A(0) = D, A(1) = A$*. Consider the* n *disks as* t *varies from* $t = 0$ *to* $t = 1$*. When* $t = 0$ *each disk contains exactly one eigenvalue. As* t *increases (in a continuous way) fom 0 to one – each disk will still contain one eigenvalue - by a continuity argument [you cannot have an eigenvalue jump suddently - from one disk to another- this would be a dicontinuous behavior]. The argument can be adapted to the case where two disks touch each other at one point (only): it is now possible to have two eigenvalues at the intersection of the disks - coming from each of the t2o disks.*