✍ 2 *Use the min-max theorem to show that* $\|A\|_2 = \sigma_1(A)$ *- the largest singular value of* A.

Solution: *This comes from the fact that:*

$$\begin{aligned}\|A\|_2^2 &= \max_{x\neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2} \\ &= \max_{x\neq 0} \frac{(Ax, Ax)}{(x, x)} \\ &= \max_{x\neq 0} \frac{(A^T Ax, A)}{(x, x)} \\ &= \lambda_{max}(A^T A) \\ &= \sigma_1^2\end{aligned}$$

□

✍ 3 *Suppose that* $A = LDL^T$ *where* L *is unit lower triangular, and* D *diagonal. How many negative eigenvalues does* A *have?*

Solution: *It has as many negative eigenvalues as there are negative entries in* D ☐

✍ 4 *Assume that* A *is tridiagonal. How many operations are required to determine the number of negative eigenvalues of* A*?*

Solution: *The rough answer is* $O(n)$ *– because an LU (and therefore LDLT) factorization costs* $O(n)$*. Based on doing the LU factorization of a triagonal matrix, a more accurate answer is* $3n$ *operations.* ☐

✍ 5 *Devise an algorithm based on the inertia theorem to compute the* i*-th eigenvalue of a tridiagonal matrix.*

Solution: *Here is a matlab script:*

```
   function [sigma] = bisect(d, b, i, tol)
%% function [sigma] = bisect(d, b, i, tol)
%% d   = diagonal of T
%% b   = co-diagonal
%% i   = compute i-th eigenvalue
%% tol = tolerance used for stopping
     b(1) = 0;
   n = length(d);
 %%-------------------- guershgorin
```

```
tmin = d(n) - abs(b(n));
tmax = d(n) + abs(b(n));
for j=1:n-1
  rho = abs(b(j)) + abs(b(j+1));
  tmin = min(tmin, d(j)-rho);
  tmax = max(tmax, d(j)+rho);
end
tol = tol*(tmax-tmin);
for iter=1:100
  sigma = 0.5*(tmin+tmax);
  count = sturm(d, b, sigma);
  if (count >= i)
    tmin = sigma;
  else
    tmax = sigma;
  end
  if (tmax - tmin) < tol
    break
  end
end
```

□

✍ 6 *What is the inertia of the matrix*

$$\begin{pmatrix} \boldsymbol{I} & \boldsymbol{F} \\ \boldsymbol{F}^T & \boldsymbol{0} \end{pmatrix}$$

where $\boldsymbol{F}$ *is* $m \times n$, *with* $n < m$, *and of full rank?*

[Hint: use a block LU factorization]

Solution: *We start with*

$$\begin{pmatrix} I & F \\ F^T & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ F^T & I \end{pmatrix} \begin{pmatrix} I & F \\ 0 & -F^T F \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ F^T & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -F^T F \end{pmatrix} \begin{pmatrix} I & F \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ F^T & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -F^T F \end{pmatrix} \begin{pmatrix} I & 0 \\ F^T & I \end{pmatrix}^T$$

This is of the form XDX^T where X is invertible.. Therefore the inertia is the same as that of the block diagonal matrix which is: m positive eigenvalues (block I) and n negative eigenvalues since $-F^TF$ is $n \times n$ and negative definite.

✍ 7 *Let $\|A_O\|_I = \max_{i\neq j} |a_{ij}|$. Show that*

$$\|A_O\|_F \leq \sqrt{n(n-1)}\|A_O\|_I$$

Solution: *This is straightforward:*

$$\|A_O\|_F^2 = \sum_{i \neq j} |a_{ij}|^2 \leq n(n-1) \max_{i \neq j} |a_{ij}|^2 = n(n-1)\|A_O\|_I^2.$$

□

✍ 8 *Use this to show convergence in the case when largest entry is zeroed at each step.*

Solution: *If we call B_k the matrix A_O after each rotation then we have according to result in the previous page and using the previous exercise:*

$$\begin{aligned} \|B_{k+1}\|_F^2 &= \|B_k\|_F^2 - 2a_{pq}^2 \\ &= \|B_k\|_F^2 - 2\|B_k\|_I^2 \\ &\leq \|B_k\|_F^2 - \frac{2}{n(n+1)}\|B_k\|_F^2 \\ &= \left[1 - \frac{2}{n(n+1)}\right] \|B_k\|_F^2 \end{aligned}$$

which shows that the norm will be decreasing by factor less than a constant that is less than one - therefore it converges to zero. □