Use the min-max theorem to show that $\|A\|_2 = \sigma_1(A) - the largest singular value of A.$

Solution: This comes from the fact that:

$$\|A\|_2^2 = \max_{x \neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2} = \max_{x \neq 0} \frac{(Ax, Ax)}{(x, x)}$$

$$= \max_{x \neq 0} \frac{(A^T A x, A)}{(x, x)}$$

$$= \lambda_{max}(A^T A)$$

$$= \sigma_1^2$$

Suppose that $A = LDL^T$ where $L$ is unit lower triangular, and $D$ diagonal. How many negative eigenvalues does $A$ have?
Solution: It has as many negative eigenvalues as there are negative entries in $D$.

4. Assume that $A$ is tridiagonal. How many operations are required to determine the number of negative eigenvalues of $A$?

Solution: The rough answer is $O(n)$ – because an LU (and therefore LDLT) factorization costs $O(n)$. Based on doing the LU factorization of a tridiagonal matrix, a more accurate answer is $3n$ operations.

5. Devise an algorithm based on the inertia theorem to compute the $i$-th eigenvalue of a tridiagonal matrix.

Solution: Here is a matlab script:

```matlab
function [sigma] = bisect(d, b, i, tol)
%% function [sigma] = bisect(d, b, i, tol)
%% d = diagonal of T
%% b = co-diagonal
%% i = compute i-th eigenvalue
%% tol = tolerance used for stopping
    b(1) = 0;
    n = length(d);
%%------------------------ guershgorin
```
\begin{verbatim}
tmin = d(n) - abs(b(n));
tmax = d(n) + abs(b(n));
for j=1:n-1
    rho = abs(b(j)) + abs(b(j+1));
    tmin = min(tmin, d(j)-rho);
    tmax = max(tmax, d(j)+rho);
end
tol = tol*(tmax-tmin);
for iter=1:100
    sigma = 0.5*(tmin+tmax);
    count = sturm(d, b, sigma);
    if (count >= i)
        tmin = sigma;
    else
        tmax = sigma;
    end
    if (tmax - tmin) < tol
        break
    end
end
\end{verbatim}

What is the inertia of the matrix 
\[
\begin{pmatrix}
I & F \\
F^T & 0
\end{pmatrix}
\]
where $F$ is $m \times n$, with $n < m$, and of full rank?
[Hint: use a block LU factorization]

Solution: We start with

\[
\begin{pmatrix}
I & F \\
F^T & 0
\end{pmatrix}
= \begin{pmatrix}
I & 0 \\
F^T & I
\end{pmatrix}
\begin{pmatrix}
I & F \\
0 & -F^T F
\end{pmatrix}
= \begin{pmatrix}
I & 0 \\
F^T & I
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & -F^T F
\end{pmatrix}
\begin{pmatrix}
I & F \\
0 & I
\end{pmatrix}
= \begin{pmatrix}
I & 0 \\
F^T & I
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & -F^T F
\end{pmatrix}
\begin{pmatrix}
I & F \\
0 & I
\end{pmatrix}

This is of the form \( XDX^T \) where \( X \) is invertible. Therefore the inertia is the same as that of the block diagonal matrix which is: \( m \) positive eigenvalues (block \( I \)) and \( n \) negative eigenvalues since \(-F^T F\) is \( n \times n \) and negative definite.

Let \( \| A_O \|_I = \max_{i \neq j} |a_{ij}| \). Show that

\[
\| A_O \|_F \leq \sqrt{n(n-1)} \| A_O \|_I
\]
Solution: This is straightforward:

\[ \| A_O \|_F^2 = \sum_{i \neq j} |a_{ij}|^2 \leq n(n - 1) \max_{i \neq j} |a_{ij}|^2 = n(n - 1) \| A_O \|_I^2. \]

\[ \square \]

Use this to show convergence in the case when largest entry is zeroed at each step.

Solution: If we call \( B_k \) the matrix \( A_O \) after each rotation then we have according to result in the previous page and using the previous exercise:

\[ \| B_{k+1} \|_F^2 = \| B_k \|_F^2 - 2a_{pq}^2 \]
\[ = \| B_k \|_F^2 - 2\| B_k \|_I^2 \]
\[ \leq \| B_k \|_F^2 - \frac{2}{n(n + 1)} \| B_k \|_F^2 \]
\[ = \left[ 1 - \frac{2}{n(n + 1)} \right] \| B_k \|_F^2 \]

which shows that the norm will be decreasing by factor less than a constant that is less than one - therefore it converges to zero. \( \square \)