✍1 *Exact solution of system*

$$\begin{pmatrix} 2 & 4 & 4 \\ 1 & 5 & 6 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix}$$

Solution: *You will find* $x = [1, 3, -2]^T$. □

✍2 *Justify the column version of Back-subsitution algorithm.*

Solution: *The system* $Ax = b$ *can be written in column form as follows:*

$$x_1 a_{:,1} + x_2 a_{:,2} + \cdots + x_n a_{:,n} = b$$

In first step we compute $x_n = b_n / a_{n,n}$. *Now move last term in left-hand side of above system* *to the right:*

$$x_1 a_{:,1} + x_2 a_{:,2} + \cdots + x_{n-1} a_{:,n-1} = b - x_n a_{:,n} \equiv b^{(1)}$$

This is a new system of n equations that has $(n-1)$ unknowns and the right-hand-side $b^{(1)}$. The last equation of this system is of the form $0 = 0$ and can therefore be ignored. Thus, we end up wih a system of size $(n-1) \times (n-1)$ that is still upper triangular and we can repeat the above argument recursively. ☐

✍ 3 *Exact operation count for GE.*

Solution:

$$\begin{aligned} T &= \sum_{k=1}^{n-1} \sum_{i=k+1}^{n} (2(n-k)+3) \\ &= \sum_{k=1}^{n-1} (2(n-k)+3)(n-k) \end{aligned}$$

$$\begin{aligned} T &= 2\sum_{k=1}^{n-1}(n-k)^2 + 3\sum_{k=1}^{n-1}(n-k) \\ &= 2\sum_{j=1}^{n-1} j^2 + 3\sum_{j=1}^{n-1} j \end{aligned}$$

In the last step we made a change of variables $j = n - k$. Now we know that $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$ and $\sum_{k=1}^{n} k = n(n+1)/2$ and so

$$\begin{aligned} T &= 2\frac{(n-1)(n)(2n-1)}{6} + 3 \times \frac{n(n-1)}{2} \\ &= .... \\ &= n(n-1)\left(\frac{2n}{3} + \frac{7}{6}\right) \end{aligned} \tag{1}$$

Finally observe the remarkable fact that the final expression (1) is always an integer (it has to be) no matter what (integer) value n takes. □

✍ 4 *Practical use: Show how to use the LU factorization to solve linear systems with the same matrix A and different b's.*

Solution: *If we have the LU factorization* $A = LU$ *available then we can solve the linear system* $Ax = b$ *by writing it as*

$$L \underbrace{(Ux)}_{y} = b$$

So we solve for y*:* $Ly = b$ *then once* y *is computed we solve for* x*:* $Ux = y$*. This involves two triangular solves at the cost of* n^2 *each instead of the* $O(n^3)$ *cost of redoing everything with Gaussian elimination.*□

✍5 *LU factorization of the matrix* $A = \begin{pmatrix} 2 & 4 & 4 \\ 1 & 5 & 6 \\ 1 & 3 & 1 \end{pmatrix}$*?*

Solution: *You will find*

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1/3 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 2 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & -7/3 \end{pmatrix} \qquad \square$$

✍6 *Determinant of* A*?*

Solution: *It is the determinant of U which is -12.*

✍7 *True or false: "Computing the LU factorization of matrix A involves more arithmetic operations than solving a linear system $Ax = b$ by Gaussian elimination".*

Solution: *The number of arithmetic operations is identical. (The LU factorization involves additional memory moves to store the factors - but these are no floating point operations)* □

✍8 *Operation count for Gauss-Jordan. Order of the cost? How does it compare with Gaussian Elimination?*

Solution: *From the notes:*

$$\begin{aligned} T &= \sum_{k=1}^{n-1} \sum_{i=1}^{n-1} [2(n-k)+3)] = \sum_{k=1}^{n-1} (n-1)[2(n-k)+3] \\ &= (n-1) \sum_{j=1}^{n-1} [2j+3] \\ &= (n-1)\left[n(n-1)+3(n-1)\right] \\ &= (n-1)^2(n+3) = (n-1)^3 + 4(n-1)^2 \end{aligned}$$

The bottom line is that the cost is $\approx n^3$ which is 50% more expensive than GE. This additional cost is not worth it in spite of the simplicity of the algorithm. For this Gauss-Jordan is seldom used in practice. □

✍ 9 *What is the matrix PA when*

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 0 & -1 & 2 \\ -3 & 4 & -5 & 6 \end{pmatrix} ?$$

Solution: *Instead of multiplying you just permute the row: row 1 in new matrix is row 3 of old* matrix, row 2 is row 1 of old matrix, etc.

$$PA = \begin{pmatrix} 9 & 0 & -1 & 2 \\ 1 & 2 & 3 & 4 \\ -3 & 4 & -5 & 6 \\ 5 & 6 & 7 & 8 \end{pmatrix}$$

✍ 10 *In the previous example where*

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>> A = [ 1 2 3 4; 5 6 7 8; 9  0 -1 2 ; -3 4 -5 6]
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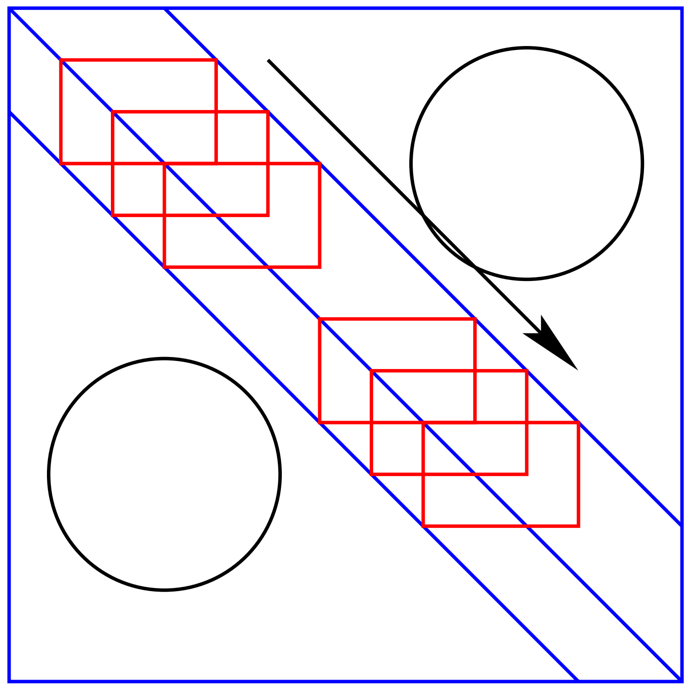
Matlab gives $det(A) = -896$. *What is* $det(PA)$*?*

Solution: *It changes sign so* $\det(PA) = 896$. *This is because the permutation* $\pi = [3, 1, 4, 2]$ *is made of 3 interchanges.* ☐

✍ 11 *Given a banded matrix with upper bandwidth* q *and lower bandwidth* p, *what is the operation count (leading term only) for solving the linear system* $Ax = b$ *with Gaussian elimination without* *pivoting? What happens when partial pivoting is used? Give the new operation count for the worst case scenario.*

Solution: *[Note: it is assumed that $p \ll n$ and $q \ll n$ but p and q are not related]. The important observation here is that Gaussian elimination without pivoting for this band matrix will operate on a rectangle: at step k only rows $k+1$ to $k+p$ are affected and columns $k+1$ to $k+q$ are affected.*



In this rectangle each entry will be modified at the cost of 2 operations (, +). Total: $2pq$ for* *each step. So Gaussian elimination without pivoting for this band matrix costs approximately* $\mathbf{2npq}$ *flops. Using band backward substitution to obtain the solution* $\boldsymbol{x}$ *costs* $\approx \mathbf{2nq}$ *flops. The total operation count (leading term only):* $\approx \mathbf{2npq + 2nq = 2nq(p+1)}$*. Note that when* $\boldsymbol{p}$ *is small the cost of susbstitution cannot be ignored.*

For the Gaussian elimination with pivoting, the upper bandwidth of the resulting matrix will be $\boldsymbol{p+q}$*. In this case, the total operation count (leading term only) becomes:* $\approx \mathbf{2np(p+q)(p+1)}$*.* □