✍1 *Show that* $\boldsymbol{\kappa(I) = 1}$ *;*

Solution: *This is obvious because for any matrix norm* $\|\boldsymbol{I}\| = \|\boldsymbol{I}^{-1}\| = 1$. $\square$

✍2 *Show that* $\boldsymbol{\kappa(A) \geq 1}$ *;*

Solution: *We have* $\|\boldsymbol{AA}^{-1}\| = \|\boldsymbol{I}\| = 1$ *therefore* $1 = \|\boldsymbol{AA}^{-1}\| \leq \|\boldsymbol{A}\| \, \|\boldsymbol{A}^{-1}\| = \boldsymbol{\kappa(A)}$ $\square$

✍5 *Show that if* $\|\boldsymbol{E}\|/\|\boldsymbol{A}\| \leq \delta$ *and* $\|\boldsymbol{e_b}\|/\|\boldsymbol{b}\| \leq \delta$ *then*

$$\frac{\|x - y\|}{\|x\|} \leq \frac{2\delta\kappa(A)}{1 - \delta\kappa(A)}$$

Solution: *From the main theorem (theorem 1) we have*

$$\frac{\|x - y\|}{\|x\|} \leq \frac{\|A^{-1}\| \, \|A\|}{1 - \|A^{-1}\| \, \|E\|} \left(\frac{\|E\|}{\|A\|} + \frac{\|e_b\|}{\|b\|} \right)$$

If $\|E\| \leq \delta$ *and* $\|e_b\|/\|b\| \leq \delta$ *then:*

$$\frac{\|x-y\|}{\|x\|} \leq \frac{\kappa(A) \times 2\delta}{1 - \|A^{-1}\| \ \|E\|}$$
$$\leq \frac{2\delta\kappa(A)}{1 - \|A^{-1}\|\|A\| \times (\|E\|/\|A\|)}$$
$$\leq \frac{2\delta\kappa(A)}{1 - \delta\kappa(A)}.$$

□

✍ 9 *Show that* $\frac{\|x-\tilde{x}\|}{\|x\|} \geq \frac{1}{\kappa(A)} \frac{\|r\|}{\|b\|}$.

Solution: *As before we start with noting that* $A(x - \tilde{x}) = b - A\tilde{x} = r$. *So:*

$$\|r\| \leq \|A\|\|x - \tilde{x}\| \rightarrow \frac{\|r\|}{\|b\|} \leq \|A\| \frac{\|x - \tilde{x}\|}{\|b\|}$$

Next from $\|x\| = \|A^{-1}b\| \leq \|A^{-1}\|\|b\|$ *we get* $\|b\| \geq \|x\|/\|A^{-1}\|$ *and so*

$$\frac{\|r\|}{\|b\|} \leq \|A\| \frac{\|x - \tilde{x}\|}{\|x\|/\|A^{-1}\|} = \kappa(A) \frac{\|x - \tilde{x}\|}{\|x\|}$$

which yields the result after dividing the 2 sides by $\kappa(A)$. □

Proof of Theorem 3

Let $D \equiv \|E\|\|y\| + \|e_b\|$ *and* $\eta \equiv \eta_{E,e_b}(y)$. *The theorem states that* $\eta = \|r\|/D$. *Proof in 2 steps.*

First: *Any* $\Delta A, \Delta b$ *pair satisfying (1) is such that* $\epsilon \geq \|r\|/D$. *Indeed from (1) we have (recall that* $r = b - Ay$*)*

$$Ay + \Delta A y = b + \Delta b \rightarrow r = \Delta A y - \Delta b \rightarrow$$

$$\|r\| \leq \|\Delta A\|\|y\| + \|\Delta b\| \leq \epsilon(\|E\|\|y\| + \|e_b\|) \rightarrow \epsilon \geq \frac{\|r\|}{D}$$

Second: *We need to show an instance where the minimum value of* $\|r\|/D$ *is reached. Take the pair* $\Delta A, \Delta b$*:*

$$\Delta A = \alpha r z^T; \quad \Delta b = \beta r \quad \text{with } \alpha = \frac{\|E\|\|y\|}{D}; \quad \beta = \frac{\|e_b\|}{D}$$

The vector z depends on the norm used - for the 2-norm: $z = y/\|y\|^2$. Here: Proof only for 2-norm

a) We need to verify that first part of (1) is satisfied:

$$
\begin{aligned}
(A + \Delta A)y &= Ay + \alpha r \frac{y^T}{\|y\|^2} y = b - r + \alpha r \\
&= b - (1-\alpha) r = b - \left(1 - \frac{\|E\| \, \|y\|}{\|E\|\|y\| + \|e_b\|}\right) r \\
&= b - \frac{\|e_b\|}{D} r = b + \beta r \quad \to
\end{aligned}
$$

$$(A + \Delta A)y = b + \Delta b \quad \leftarrow \text{ The desired result}$$

Finally: b) Must now verify that $\|\Delta A\| = \eta \|E\|$ and $\|\Delta b\| = \eta \|e_b\|$. *Exercise:* Show that $\|uv^T\|_2 = \|u\|_2 \|v\|_2$

$$\|\Delta A\| = \frac{|\alpha|}{\|y\|^2} \|ry^T\| = \frac{\|E\| \, \|y\|}{D} \frac{\|r\| \, \|y\|}{\|y\|^2} = \eta \|E\|$$

$$\|\Delta b\| = |\beta| \|r\| = \frac{\|e_b\|}{D} \|r\| = \eta \|e_b\| \qquad QED$$