✍1 *Show that each* A_k *[*$A(1:k,1:k)$ *in matlab notation] is SPD.*

Solution: *Let* x *be any vector in* $\mathbb{R}^k$ *and consider the vector* y *of* $\mathbb{R}^n$ *obtained by stacking* x *followed by* $n-k$ *zeros. Then it can be easily seen that :* $(A_k x, x) = (Ay, y)$ *and since* A *is SPD then* $(Ay, y) > 0$ *and therefore* $(A_k x, x) > 0$ *for any* x *in* $\mathbb{R}^k$*. Hence* A_k *is SPD.* □

✍2 *Consequence* $\det(A_k) > 0$

Solution: *This is because the determinant is the product of the eigenvalues which are real positive (see notes).* □

✍3 *If* A *is SPD then for any* $n \times k$ *matrix* X *of rank* k*, the matrix* $X^T A X$ *is SPD.*

Solution: *For any* $v \in \mathbb{R}^k$ *we have* $(X^T A X v, v) = (AXv, Xv)$*. In addition, since* X *is of full rank, then* Xv *cannot be zero if* v *is nonzero. Therefore we have* $(AXv, Xv) > 0$*.* □

✍ 4 *Show that if* $\boldsymbol{A}^T = \boldsymbol{A}$ *and* $(\boldsymbol{Ax}, \boldsymbol{x}) = \boldsymbol{0}\ \forall \boldsymbol{x}$ *then* $\boldsymbol{A} = \boldsymbol{0}$.

Solution: *The condition implies that for all* $\boldsymbol{x}, \boldsymbol{y}$ *we have* $(\boldsymbol{A}(\boldsymbol{x}+\boldsymbol{y}), \boldsymbol{x}+\boldsymbol{y}) = \boldsymbol{0}$. *Now expand this as:* $(\boldsymbol{Ax}, \boldsymbol{x}) + (\boldsymbol{Ay}, \boldsymbol{y}) + 2(\boldsymbol{Ax}, \boldsymbol{y}) = \boldsymbol{0}$ *for all* $\boldsymbol{x}, \boldsymbol{y}$ *which shows that* $(\boldsymbol{Ax}, \boldsymbol{y}) = \boldsymbol{0}\ \forall\ \boldsymbol{x}, \boldsymbol{y}$. *This implies that* $\boldsymbol{A} = \boldsymbol{0}$ *(e.g. take* $\boldsymbol{x} = \boldsymbol{e}_j, \boldsymbol{y} = \boldsymbol{e}_i$*)...* ☐

✍ 5 *Show: A nonzero matrix* $\boldsymbol{A}$ *is indefinite iff* $\exists\ \boldsymbol{x}, \boldsymbol{y} : (\boldsymbol{Ax}, \boldsymbol{x})(\boldsymbol{Ay}, \boldsymbol{y}) < \boldsymbol{0}$.

Solution:

[$\leftarrow$] *Trivial. The matrix cant be PSD or NSD under the conditon*

[$\rightarrow$] *Need to prove: If* $\boldsymbol{A}$ *is indefinite then there exist such that* $\boldsymbol{x}, \boldsymbol{y} : (\boldsymbol{Ax}, \boldsymbol{x})(\boldsymbol{Ay}, \boldsymbol{y}) < \boldsymbol{0}$. *Assume contrary is true, i.e.,* $\forall \boldsymbol{x}, \boldsymbol{y} (\boldsymbol{Ax}, \boldsymbol{x})(\boldsymbol{Ay}, \boldsymbol{y}) \geq \boldsymbol{0}$. *There is at least one* $\boldsymbol{x}_0$ *such that* $(\boldsymbol{Ax}_0, \boldsymbol{x}_0)$ *is nonzero, otherwise* $\boldsymbol{A} = \boldsymbol{0}$ *from previous question. Assume* $(\boldsymbol{Ax}_0, \boldsymbol{x}_0) > \boldsymbol{0}$. *Then* $\forall \boldsymbol{y} (\boldsymbol{Ax}_0, \boldsymbol{x}_0)(\boldsymbol{Ay}, \boldsymbol{y}) \geq \boldsymbol{0}$. *which implies* $\forall \boldsymbol{y} : (\boldsymbol{Ay}, \boldsymbol{y}) \geq \boldsymbol{0}$, *i.e.,* $\boldsymbol{A}$ *is positive semi-definite. This contradicts the assumption that* $\boldsymbol{A}$ *is neither positive nor negative semi-defininte* ☐

✍ 6 *The (standard) LU factorization of an SPD matrix* $\boldsymbol{A}$ *exists.*

Solution: *This is an immediate consequence of the main theorem on existence (Lec. notes. set #5) and Exercise 1 in this set which showed that* $\det(A_k) > 0$ *for* $k = 1, \cdots, n$. ☐

Example:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 9 \end{pmatrix}$$

✍ 7 *Is* A *symmetric positive definite?*

Solution: *Answer is yes because* $\det(A_k) > 0$ *for* $k = 1, 2, 3$. ☐

✍ 8 *What is the* LDL^T *factorization of* A *?*

Solution: *The LU factorizatis is:*

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 1/2 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

Therefore $A = LDL^T$ *where* L *is as given above and*

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad \square$$

✍ 9 *What is the Cholesky factorization of* A *?*

Solution: *From the above LDLT factorization we have* $A = GG^T$ *with*

$$G = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 1 & 2 \end{pmatrix} \qquad \square$$

Gradient of $\psi(x) = (Ax, x)$

In practice exercise # 6 it is asked: Let A be symmetric and $\psi(x) = (Ax, x)$. What is the partial derivative $\frac{\partial \psi(x)}{\partial x_k}$? What is the gradient of ψ?

Solution: *First note that*

$$\psi(x) = \sum_{i=1}^{n} x_i \left[\sum_{j=1}^{n} a_{ij} x_j \right]$$

and so, using basic rules for derivatives of products:

$$\begin{aligned} \frac{\partial \psi(x)}{\partial x_k} &= \sum_{i=1}^{n} \frac{\partial x_i}{\partial x_k} \left[\sum_{j=1}^{n} a_{ij} x_j \right] + \sum_{i=1}^{n} x_i \left[\frac{\partial x_i}{\partial x_k} \sum_{j=1}^{n} a_{ij} x_j \right] \\ &= \sum_{j=1}^{n} a_{kj} x_j + \sum_{i=1}^{n} x_i a_{ik} \\ &= 2 \sum_{j=1}^{n} a_{kj} x_j \end{aligned}$$

which is nothing but twice the k-th component of Ax or $\frac{\partial \psi(x)}{\partial x_k} = 2(Ax)_k$. Therefore the gradient *of* ψ *is*

$$\nabla\psi(x) = 2Ax.$$

A somewhat simpler solution for finding the gradient is to expand $\psi(x+\delta) = (A(x+\delta), (x+\delta)) = \dots$ *and to write that the linear term should be of the form* $[\nabla\psi]^T\,\delta$. $\square$