✍1 *Show that* $(I - \beta v v^T)x = \alpha e_1$ *when* $v = x - \alpha e_1$ *and* $\alpha = \pm\|x\|_2$.

Solution: *Equivalent to showing that*

$$x - (\beta x^T v)v = \alpha e_1 \quad \text{i.e.,} \quad x - \alpha e_1 = (\beta x^T v)v$$

but recall that $v = x - \alpha e_1$ *so we need to show that*

$$\beta x^T v = 1 \quad \text{i.e., that} \quad \frac{2}{\|x - \alpha e_1\|_2^2}(x^T v) = 1$$

- *Denominator* $= \|x\|_2^2 + \alpha^2 - 2\alpha e_1^T x = 2(\|x\|_2^2 - \alpha e_1^T x)$
- *Numerator* $= 2x^T v = 2x^T(x - \alpha e_1) = 2(\|x\|_2^2 - \alpha x^T e_1)$

Numerator/ Denominator = 1. □

✍2 *Cost of Householder QR?*

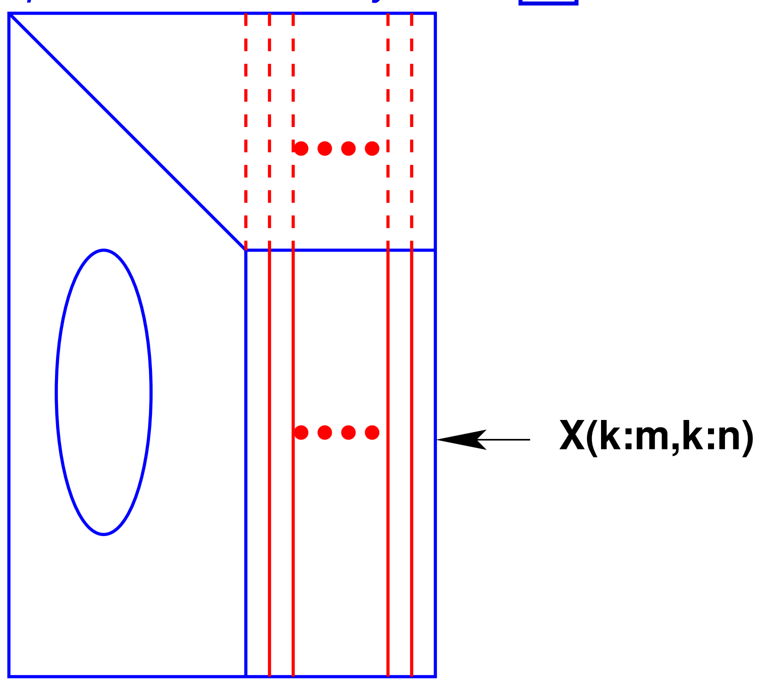
Solution: *Look at the algorithm: each step works in rectangle* $X(k:m,k:n)$. *Step* k *: twice* $2(m-k+1)(n-k+1)$

$$
\begin{aligned}
T(n) &= \sum_{k=1}^{n} 4(m-k+1)(n-k+1) \\
&= 4\sum_{k=1}^{n}[(m-n)+(n-k+1)](n-k+1) \\
&= 4[(m-n)*\frac{n(n+1)}{2}+\frac{n(n+1)(2n+1)}{6}] \\
&\approx (m-n)*2n^2+4n^3/3 \\
&= 2mn^2-\frac{2}{3}n^3
\end{aligned}
$$

□

✍ 3 *Suppose you know the norms of each column of* X *at the start. What happens to each of the norms of* $X(2:m,j)$ *for* $j=2,\cdots,n$*? Generalize this to step* k *and obtain a procedure to inexpensively compute the desired norms at each step.*

Solution: *The trick that is used is that* the 2-norm of each column does not change thoughout the algorithm. *This is simple to see because each column is multiplied by a Householder transformation* P_k *at each step. These Householder transformations are unitary and preserve the length. The square of the 2-norm of* $X(k:n,j)$ *(solid red lines in Figure) is the original square of the 2-norm of* $X(k:n,j)$ *minus the square of the 2-norm of* $X(1:k-1,j)$ *(dashed red lines in Figure). (solid red lines in Figure) In order to* update $\|X(k:n,j)\|^2$ *– all we have to do is subtract* $|X(k-1,j)|^2$ *at each step* k*. This costs very little.* □



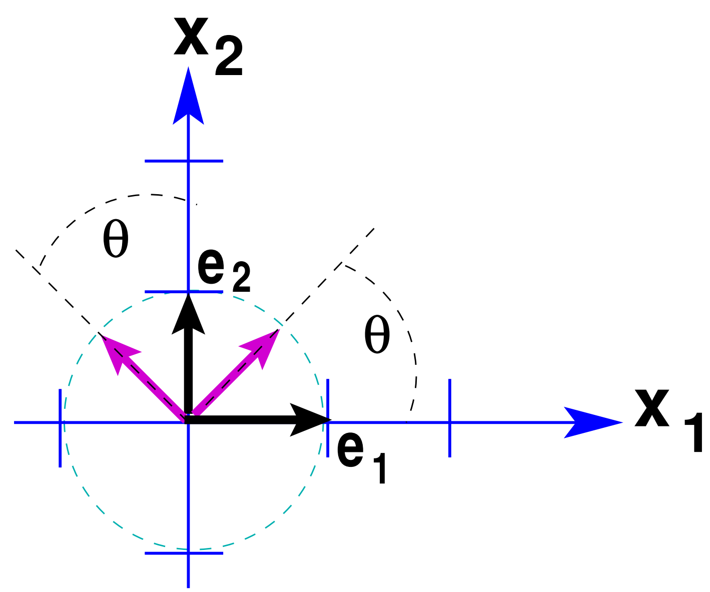
✍ 4 *Consider the mapping that sends any point $\boldsymbol{x}$ in $\mathbb{R}^2$ into a point $\boldsymbol{y}$ in $\mathbb{R}^2$ that is rotated from $\boldsymbol{x}$ by an angle $\boldsymbol{\theta}$. Find the matrix representing the mapping. [Hint: observe how the canonical basis is transformed.] Show an illustration. What is the mapping correspoding to an angle $-\boldsymbol{\theta}$?*

Solution: *The vector* $\boldsymbol{e}_1 = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}$ *is transformed to* $\begin{pmatrix} \cos\boldsymbol{\theta} \\ \sin\boldsymbol{\theta} \end{pmatrix}$. *The vector* $\boldsymbol{e}_2 = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}$ *is transformed to* $\begin{pmatrix} -\sin\boldsymbol{\theta} \\ \cos\boldsymbol{\theta} \end{pmatrix}$.

These are the first and second columns of the mapping! So the matrix representing the rotation is

$$\boldsymbol{R}_\theta = \begin{pmatrix} \cos\boldsymbol{\theta} & -\sin\boldsymbol{\theta} \\ \sin\boldsymbol{\theta} & \cos\boldsymbol{\theta} \end{pmatrix}$$

An illustration is shown in the figure.



A Givens rotation performs a rotation of angle $-\boldsymbol{\theta}$. □