✍ 1 *How would you get an orthonormal basis of X?*

Solution: *Just take any basis and orthonormalize it with Gram-Schmidt of Householder QR.* □

✍ 2 *Show how you can get a decomposition in which C is lower (or upper) triangular, from the above factorization.*

Solution: *You first get any factorization in the form shown in Page 9-7 – Then to get an upper triangular C you use the QR factorization $C = QR$. Then U is replaced by*

$$U_{new} = U \times \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix}$$

and C is replaced by R. To get a lower triangular C you can use the same trick applied to A^T and transpose the final result. □

✍ 3 *How can you get the ULV decomposition by using only the Householder QR factorization* *(possibly with pivoting)?*

Solution: *You first get the Householder QR factorization* $A = Q_1 R_1$ *of the matrix* A. *The second step is to perform a Householder QR factorization of the matrix* R_1^T, *so you will get:* $R_1^T = Q_2 R_2$. *The final step is to write:*

$$A = Q_1 * R_2^T * Q_2^T \equiv URV^T$$

where $U = Q_1 \in \mathbb{R}^{m\times m}$; $V = Q_2 \in \mathbb{R}^{n\times n}$; $R = R_2 \in \mathbb{R}^{m\times n}$ □

✍ 4 *In the proof of the SVD decomposition, define* U, V *as single Householder reflectors.*

Solution: *We deal with* U *only [proceed similarly with* V*]. We need a matrix* $P = I - 2ww^T$ *such that the first column of* A *is* u_1 *and all columns are orthonormal. The second requirement is satisfied by default since* P *is unitary. Note that if* P *is available we will have* $Pu_1 = e_1$ *because* $P^2 = I$. *Therefore, the wanted* w *is simply the vector that transforms the vector* u_1 *into* αe_1.
... □

✍ 5 *How can you obtain the thin SVD from the QR factorization of* A *and the SVD of an* $n \times n$

matrix?

Solution: *We first get the thin QR factorization of* $\boldsymbol{A}$, *namely* $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$ *where* $\boldsymbol{Q} \in \mathbb{R}^{m\times n}$ *and* $\boldsymbol{R} \in \mathbb{R}^{n\times n}$*; Then we can get the SVD* $\boldsymbol{R} = \boldsymbol{U_R}\boldsymbol{\Sigma}\boldsymbol{V_R^T}$ *of* $\boldsymbol{R}$ *and this yields:*

$$\boldsymbol{A} = \boldsymbol{Q} \times \boldsymbol{U_R}\boldsymbol{\Sigma_R}\boldsymbol{V_R^T} \rightarrow \boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V^T}, \quad \textit{with} \quad \boldsymbol{U} = \boldsymbol{Q} \times \boldsymbol{U_R}; \quad \boldsymbol{\Sigma} = \boldsymbol{\Sigma_R}; \quad \boldsymbol{V} = \boldsymbol{V_R}.$$