# ERROR AND SENSITIVITY ANALYSIS FOR SYSTEMS OF LINEAR EQUATIONS

- **Conditioning of linear systems.**
- **Estimating errors for solutions of linear systems**
- **(Normwise) Backward error analysis**
- **Estimating condition numbers ..**

## *Perturbation analysis for linear systems ($Ax = b$)*

Question addressed by perturbation analysis: determine the variation of the solution $\boldsymbol{x}$ when the data, namely $\boldsymbol{A}$ and $\boldsymbol{b}$, undergoes small variations. Problem is Ill-conditioned if small variations in data cause very large variation in the solution.

*Setting:*

➤ We perturb $\boldsymbol{A}$ into $\boldsymbol{A} + \boldsymbol{E}$ and $\boldsymbol{b}$ into $\boldsymbol{b} + \boldsymbol{e_b}$. Can we bound the resulting change (perturbation) to the solution?

*Preparation:* We begin with a lemma for a simple case

## *Rigorous norm-based error bounds*

*LEMMA:* If $\|E\| < 1$ then $I - E$ is nonsingular and

$$\|(I - E)^{-1}\| \le \frac{1}{1 - \|E\|}$$

*Proof* is based on following 5 steps

a) Show: If $\|E\| < 1$ then $I - E$ is nonsingular

b) Show: $(I - E)(I + E + E^2 + \cdots + E^k) = I - E^{k+1}$.

c) From which we get:

$$(I - E)^{-1} = \sum_{i=0}^{k} E^i + (I - E)^{-1} E^{k+1} \rightarrow$$

d) $(I - E)^{-1} = \lim_{k\to\infty} \sum_{i=0}^{k} E^i$. We write this as

$$(I - E)^{-1} = \sum_{i=0}^{\infty} E^i$$

e) Finally:

$$\begin{aligned}\|(I - E)^{-1}\| &= \left\| \lim_{k\to\infty} \sum_{i=0}^{k} E^i \right\| = \lim_{k\to\infty} \left\| \sum_{i=0}^{k} E^i \right\| \\ &\le \lim_{k\to\infty} \sum_{i=0}^{k} \|E^i\| \le \lim_{k\to\infty} \sum_{i=0}^{k} \|E\|^i \\ &\le \frac{1}{1 - \|E\|}\end{aligned}$$

➤ Can generalize result:

*LEMMA:* If $A$ is nonsingular and $\|A^{-1}\|\ \|E\| < 1$ then $A + E$ is non-singular and

$$\|(A+E)^{-1}\| \leq \frac{\|A^{-1}\|}{1-\|A^{-1}\|\ \|E\|}$$

➤ Proof is based on relation $A + E = A(I + A^{-1}E)$ and use of previous lemma.

➤ Now we can prove the main theorem:

THEOREM 1: Assume that $(A + E)y = b + e_b$ and $Ax = b$ and that $\|A^{-1}\|\|E\| < 1$. Then $A + E$ is nonsingular and

$$\frac{\|x-y\|}{\|x\|} \leq \frac{\|A^{-1}\|\ \|A\|}{1-\|A^{-1}\|\ \|E\|}\left(\frac{\|E\|}{\|A\|}+\frac{\|e_b\|}{\|b\|}\right)$$

Proof: From $(A + E)y = b + e_b$ and $Ax = b$ we get $(A + E)(y - x) = e_b - Ex$. Hence:

$$y - x = (A + E)^{-1}(e_b - Ex)$$

Taking norms $\rightarrow$ $\|y - x\| \le \|(A + E)^{-1}\| \, [\|e_b\| + \|E\| \|x\|]$
Dividing by $\|x\|$ and using result of lemma

$$\begin{aligned}
\frac{\|y - x\|}{\|x\|} &\le \|(A + E)^{-1}\| \, [\|e_b\|/\|x\| + \|E\|] \\
&\le \frac{\|A^{-1}\|}{1 - \|A^{-1}\| \|E\|} [\|e_b\|/\|x\| + \|E\|] \\
&\le \frac{\|A^{-1}\| \|A\|}{1 - \|A^{-1}\| \|E\|} \left[ \frac{\|e_b\|}{\|A\| \|x\|} + \frac{\|E\|}{\|A\|} \right]
\end{aligned}$$

Result follows by using inequality $\|A\| \|x\| \ge \|b\|$.... QED

The quantity $\boxed{\kappa(A) = \|A\| \, \|A^{-1}\|}$ is called the condition number of the linear system with respect to the norm $\|.\|$. When using the $p$-norms we write:

$$\kappa_p(A) = \|A\|_p \|A^{-1}\|_p$$

➤ Note: $\kappa_2(A) = \sigma_{max}(A)/\sigma_{min}(A)$ = ratio of largest to smallest singular values of $A$. Allows to define $\kappa_2(A)$ when $A$ is not square.

➤ Determinant *is not* a good indication of sensitivity

➤ Small eigenvalues *do not* always give a good indication of poor conditioning.

***Example:*** Consider, for a large $\alpha$, the $n \times n$ matrix

$$A = I + \alpha e_1 e_n^T$$

➤ Inverse of $A$ is : $A^{-1} = I - \alpha e_1 e_n^T$ ➤ For the $\infty$-norm we have

$$\|A\|_\infty = \|A^{-1}\|_\infty = 1 + |\alpha|$$

so that $$\kappa_\infty(A) = (1 + |\alpha|)^2.$$

➤ Can give a very large condition number for a large $\alpha$ – but all the eigenvalues of $A$ are equal to one.

✍1 Show that $\kappa(I) = 1$ ;

✍2 Show that $\kappa(A) \geq 1$ ;

✍3 Show that $\kappa(A) = \kappa(A^{-1})$

✍4 Show that for $\alpha \neq 0$, we have $\kappa(\alpha A) = \kappa(A)$

Simplification when $e_b = 0$ :

$$\frac{\|x - y\|}{\|x\|} \le \frac{\|A^{-1}\| \; \|E\|}{1 - \|A^{-1}\| \; \|E\|}$$

Simplification when $E = 0$ :

$$\frac{\|x - y\|}{\|x\|} \le \|A^{-1}\| \; \|A\| \frac{\|e_b\|}{\|b\|}$$

➤ Slightly less general form: Assume that $\|E\|/\|A\| \le \delta$ and $\|e_b\|/\|b\| \le \delta$ and $\delta\kappa(A) < 1$ then

$$\frac{\|x - y\|}{\|x\|} \le \frac{2\delta\kappa(A)}{1 - \delta\kappa(A)}$$

✍5 Show the above result

## *Another common form:*

THEOREM 2: Let $(A + \Delta A)y = b + \Delta b$ and $Ax = b$ where $\|\Delta A\| \leq \epsilon\|E\|$, $\|\Delta b\| \leq \epsilon\|e_b\|$, and assume that $\epsilon\|A^{-1}\|\|E\| < 1$. Then

$$\frac{\|x - y\|}{\|x\|} \leq \frac{\epsilon \, \|A^{-1}\| \, \|A\|}{1 - \epsilon\|A^{-1}\| \, \|E\|} \left( \frac{\|e_b\|}{\|b\|} + \frac{\|E\|}{\|A\|} \right)$$

➤ Results to be seen later are of this type.

## *Normwise backward error*

- We solve $Ax = b$ and find an approximate solution $y$

*Question:* Find smallest perturbation to apply to $A, b$ so that *exact* solution of perturbed system is $y$

## *Normwise backward error in just A or b*

**Suppose we model entire perturbation in RHS $b$.**

- Let $r = b - Ay$ be the residual.
  Then $y$ satisfies $Ay = b + \Delta b$ with $\Delta b = -r$ exactly.
- The relative perturbation to the RHS is $\frac{\|r\|}{\|b\|}$.

**Suppose we model entire perturbation in matrix $A$.**

- Then $y$ satisfies $\left(A + \frac{ry^T}{y^Ty}\right) y = b$
- The relative perturbation to the matrix is

$$\left\|\frac{ry^T}{y^Ty}\right\|_2 / \|A\|_2 = \frac{\|r\|_2}{\|A\| \, \|y\|_2}$$

## *Normwise backward error in both A & b*

For a given $y$ and given perturbation directions $E, e_b$, we define the Normwise backward error:

$$\eta_{E,e_b}(y) = \min\{\epsilon \mid (A + \Delta A)y = b + \Delta b; \text{ where } \Delta A, \Delta b \text{ satisfy: } \|\Delta A\| \le \epsilon\|E\|; \text{ and } \|\Delta b\| \le \epsilon\|e_b\|\}$$

In other words $\eta_{E,e_b}(y)$ is the smallest $\epsilon$ for which

$$(1)\begin{cases} (A + \Delta A)y = & b + \Delta b; \\ \|\Delta A\| \le \epsilon\|E\|; & \|\Delta b\| \le \epsilon\|e_b\| \end{cases}$$

➤ $\boldsymbol{y}$ is given (a computed solution). $\boldsymbol{E}$ and $\boldsymbol{e_b}$ to be selected (most likely 'directions of perturbation for $\boldsymbol{A}$ and $\boldsymbol{b}$').

➤ Typical choice: $\boldsymbol{E} = \boldsymbol{A}$, $\boldsymbol{e_b} = \boldsymbol{b}$

✍6 Explain why this is not unreasonable

Let $\boldsymbol{r} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{y}$. Then we have:

THEOREM 3: $\boldsymbol{\eta_{E,e_b}(y) = \frac{\|r\|}{\|E\|\|y\|+\|e_b\|}}$

Normwise backward error is for case $\boldsymbol{E} = \boldsymbol{A}, \boldsymbol{e_b} = \boldsymbol{b}$:

$$\boldsymbol{\eta_{A,b}(y) = \frac{\|r\|}{\|A\|\|y\|+\|b\|}}$$

✍7 Show how this can be used in practice as a means to stop some iterative method which computes a sequence of approximate solutions to $Ax = b$.

✍8 Consider the $6 \times 6$ Vandermonde system $Ax = b$ where $a_{ij} = j^{2(i-1)}$, $b = A * [1, 1, \cdots, 1]^T$. We perturb $A$ by $E$, with $|E| \leq 10^{-10}|A|$ and $b$ similarly and solve the system. Evaluate the backward error for this case. Evaluate the forward bound provided by Theorem 2. Comment on the results.

## *Estimating condition numbers.*

- Often we just want to get a lower bound for condition number [it is 'worse than ...']
- We want to estimate $\|A\| \, \|A^{-1}\|$.
- The norm $\|A\|$ is usually easy to compute but $\|A^{-1}\|$ is not.
- We want: Avoid the expense of computing $A^{-1}$ explicitly.

*Idea:*

- Select a vector $v$ so that $\|v\| = 1$ but $\|Av\| = \tau$ is small.
- Then: $\|A^{-1}\| \geq 1/\tau$ (show why) and:

$$\kappa(A) \geq \frac{\|A\|}{\tau}$$

➤ Condition number worse than $\|A\|/\tau$ .

➤ Typical choice for $v$: choose $[\cdots \pm 1 \cdots]$ with signs chosen on the fly during back-substitution to maximize the next entry in the solution, based on the upper triangular factor from Gaussian Elimination.

➤ Similar techniques used to estimate condition numbers of large matrices in matlab.

## *Condition numbers and near-singularity*

➤ $1/\kappa \approx$ relative distance to nearest singular matrix.

Let $A, B$ be two $n \times n$ matrices with $A$ nonsingular and $B$ singular. Then

$$\frac{1}{\kappa(A)} \leq \frac{\|A - B\|}{\|A\|}$$

Proof: $B$ singular $\rightarrow$ $\exists\, x \neq 0$ such that $Bx = 0$.

$$\begin{aligned}\|x\| &= \|A^{-1}Ax\| \leq \|A^{-1}\|\ \|Ax\| = \|A^{-1}\| \|(A - B)x\| \\ &\leq \|A^{-1}\|\ \|A - B\| \|x\|\end{aligned}$$

Divide both sides by $\|x\| \times \kappa(A) = \|x\| \|A\|\ \|A^{-1}\|$ ➤ result. QED.

## *Example:*

$$\text{let } A = \begin{pmatrix} 1 & 1 \\ 1 & 0.99 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Then $\frac{1}{\kappa_1(A)} \le \frac{0.01}{2}$ ➤ $\kappa_1(A) \ge \frac{2}{0.01} = 200.$

➤ It can be shown that (Kahan)

$$\frac{1}{\kappa(A)} = \min_B \left\{ \frac{\|A - B\|}{\|A\|} \quad | \quad \det(B) = 0 \right\}$$

## *Estimating errors from residual norms*

Let $\tilde{x}$ an approximate solution to system $Ax = b$ (e.g., computed from an iterative process). We can compute the residual norm:

$$\|r\| = \|b - A\tilde{x}\|$$

Question: How to estimate the error $\|x - \tilde{x}\|$ from $\|r\|$?

➤ One option is to use the inequality

$$\frac{\|x-\tilde{x}\|}{\|x\|} \leq \kappa(A) \frac{\|r\|}{\|b\|}.$$

➤ We must have an estimate of $\kappa(A)$.

## *Proof of inequality.*

First, note that $A(x - \tilde{x}) = b - A\tilde{x} = r$. So:

$$\|x - \tilde{x}\| = \|A^{-1}r\| \leq \|A^{-1}\| \, \|r\|$$

Also note that from the relation $b = Ax$, we get

$$\|b\| = \|Ax\| \leq \|A\| \, \|x\| \quad \rightarrow \quad \|x\| \geq \frac{\|b\|}{\|A\|}$$

Therefore,

$$\frac{\|x - \tilde{x}\|}{\|x\|} \leq \frac{\|A^{-1}\| \, \|r\|}{\|b\|/\|A\|} = \kappa(A)\frac{\|r\|}{\|b\|} \qquad \square$$

✍9 Show that

$$\frac{\|x-\tilde{x}\|}{\|x\|} \geq \frac{1}{\kappa(A)} \frac{\|r\|}{\|b\|}.$$