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Convergence for the classical Jacobi [where largest entry is eliminated each time]

Recall:

$\| B_o \|_F^2 = \| A_o \|_F^2 - 2\, a(p,q)^2$

$|a(p,q)| = \| vec(A_o) \|_\infty$ number of entries is n(n-1)

$\| A_o \|_F^2 = \| vec(A_o) \|_2^2 \leq n(n-1) \| vec(A_o) \|_\infty$

recall in R^n: $|| x ||_2 \leq \sqrt{n}\, || x ||_\infty$

→

$\| B_o \|_F^2 = \| A_o \|_F^2 - 2\, a(p,q)^2 \leq \| A_o \|_F^2 - \| A_o \|_F^2/(n(n-1))$

$\| B_o \|_F^2 \leq [1 - 1/(n(n-1))] \| A_o \|_F^2$

where $a(p,q) = \max_{i \neq j} |a(i,j)| = \| vec(A_o) \|_\infty$

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Proof of min-max characterization theorem

$$\lambda_k = \max_{\dim(S)=k} \; \min_{x \in S,\, x \neq 0} \; (Ax,x) / (x,x)$$

Also:

$$\lambda_k = \max_{\dim(S)=k} \; \min_{x \in S, \|x\|=1} \; (Ax,x)$$

(Ax,x)

Let $U = [u_1, \ldots, u_n]$ be an orthonormal basis of eigenvectors

$A = U D U^H$ $D = \text{diag}[\lambda_1, \ldots, \lambda_n]$

write x in eigenbasis as $x = U y$

$(Ax,x) = (U D U^H Uy, Uy) = (U D y, Uy) = (D y, U^H Uy) = (Dy,y)$

$= \sum \lambda_i y_i^2$

a) Let S be any subs. of dim k.
   and $W = \text{span} \{u_k, \ldots, u_n\}$ [number of vectors: n-k+1]

   $S \cap W$ cannot be the zero set
   $S \cap W$ must contain a nonzero vector. Let x_v be that vector
   assume that $||x_v||_2 = 1$

$x_v = \sum y_i \; u_i \rightarrow$

$$(A\,x_v,\, x_v) = \sum_{i=k}^{n} \lambda_i \, y_i^2 \quad \text{each } \lambda_i \text{ is} \le \lambda_k$$

$$(A\,x_v,\, x_v) = \sum \lambda_i \, y_i^2 \le \lambda_k$$

therefore $\min_{x \in S, ||x||=1} (Ax,x) \le \lambda_k$ [For any S]

2) Need to show that one particular S reaches the bound
Let
$S_0 = \text{span}\{u_1, u_2, ..., u_k\}$ [dim = k]

consider : $\min_{x \in S_0, ||x||=1}$

$$(Ax,x) = \sum_{i=1}^{k} \lambda_i \, y_i^2 \ge \lambda_k$$ – and min is reached for u_k

$(Au_k, u_k) = \lambda_k$

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2-norm of A :

$|| A ||^2 = \max_x || Ax||^2 / || x||^2 = \max (Ax,Ax)/(x,x)$

$= \max (A^T Ax,x)/(x,x) = \lambda_{max} (A^TA) = \sigma_1^2$ □

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Practice exercise 11. [interlacing property]

$$A = \begin{vmatrix} B & c \\ c^T & d \end{vmatrix} \qquad \begin{matrix} y \\ 0 \end{matrix} \quad U^H \; U$$

$$\lambda_k = \max_{\dim(S)=k} \; \min_{x \in S, \|x\|=1} (Ax,x)$$

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Let $x = [y; 0]$ where $y \in \mathbb{R}^{n-1}$

$Ax = [\; By \;;\; c^T y \;]$

$(Ax,x) = (By,y)$

Let S_0 be the subspace consisting of vectors of the form x=[y;0] where $y \in S$

$\dim(S) = \dim(S_0)$

$$\lambda_k = \max_{S,\dim(S)=k} \; \min_{x \in S,\|x\|=1} (Ax,x)$$

$$\lambda_k \geq \max_{S_0,\dim(S_0)=k} \; \min_{x \in S_0,\|x\|=1} (Ax,x)$$

$$x=[y,0] \;\rightarrow\; (Ax,x) = (By,y)$$

$$\lambda_k \geq \max_{S,\dim(S)=k} \; \min_{y \in S,\|y\|=1} (By,y) = \mu_k$$

$$\lambda_k = \min_{S,\dim(S)=n-(k-1)} \; \max_{x \in S,\|x\|=1} (Ax,x)$$

$\mathrm{codim}(S)$ in $\mathbb{R}^n = n-\dim(S)$

$$\lambda_k = \min_{S,\dim(S)=n-(k-1)} \; \max_{x \in S,\|x\|=1} (Ax,x)$$

$$\leq \min_{S_0,\dim(S_0)=n-(k-1)} \; \max_{x \in S_0,\|x\|=1} (Ax,x)$$

$$\leq \min_{S,\dim(S)=n-1-(k-1)+1} \; \max_{y \in S,\|x\|=1} (By,y) = \mu_{k-1}$$

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Remember that B is of size n-1 (!)

$\dim(S) = n-k+1 = n-1 -(k-1) +1$

$\lambda_k \geq \mu_k$

$\lambda_{k+1} \leq \mu_k \leq \lambda_k$

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Courant characterization

$$\lambda_1 = \max_{||x||=1} (Ax,x)$$

$$\lambda_2 = \max_{x \perp u_1} (Ax,x)$$

$$\lambda_k = \max_{x \perp u_1,u_2,\ \ldots,\ u_{k-1}} (Ax,x)$$

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