ERROR AND SENSITIVITY ANALYSIS FOR SYSTEMS OF LINEAR EQUATIONS

- Conditioning of linear systems.
- Estimating errors for solutions of linear systems
- (Normwise) Backward error analysis
- Estimating condition numbers ...

**Perturbation analysis for linear systems** \((Ax = b)\)

Question addressed by perturbation analysis: determine the variation of the solution \(x\) when the data, namely \(A\) and \(b\), undergoes small variations. Problem is ill-conditioned if small variations in data cause very large variation in the solution.

**Setting:**

- We perturb \(A\) into \(A + E\) and \(b\) into \(b + e_b\). Can we bound the resulting change (perturbation) to the solution?

**Preparation:** We begin with a lemma for a simple case

**Rigorous norm-based error bounds**

**LEMMA:** If \(\|E\| < 1\) then \(I - E\) is nonsingular and

\[
\|(I - E)^{-1}\| \leq \frac{1}{1 - \|E\|}
\]
Proof is based on following 5 steps

a) Show: If \( \|E\| < 1 \) then \( I - E \) is nonsingular

b) Show: \( (I - E)(I + E + E^2 + \cdots + E^k) = I - E^{k+1} \).

c) From which we get:
\[
(I - E)^{-1} = \sum_{i=0}^{k} E^i + (I - E)^{-1}E^{k+1} \rightarrow
\]

\[
(I - E)^{-1} = \lim_{k \to \infty} \sum_{i=0}^{k} E^i \]

We write this as \( (I - E)^{-1} = \sum_{i=0}^{\infty} E^i \).

d) \( (I - E)^{-1} = \lim_{k \to \infty} \sum_{i=0}^{k} E^i \). We write this as
\[
(I - E)^{-1} = \sum_{i=0}^{\infty} E^i
\]

e) Finally:
\[
\| (I - E)^{-1} \| = \left\| \lim_{k \to \infty} \sum_{i=0}^{k} E^i \right\| = \lim_{k \to \infty} \left\| \sum_{i=0}^{k} E^i \right\|
\]
\[
\leq \lim_{k \to \infty} \sum_{i=0}^{k} \| E^i \| \leq \lim_{k \to \infty} \sum_{i=0}^{k} \| E \|^i
\]
\[
\leq \frac{1}{1 - \|E\|}
\]

Can generalize result:

**LEMMA:** If \( A \) is nonsingular and \( \| A^{-1} \| \| E \| < 1 \) then \( A + E \) is non-singular and
\[
\| (A + E)^{-1} \| \leq \frac{\| A^{-1} \|}{1 - \| A^{-1} \| \| E \|}
\]

Proof is based on relation \( A + E = A(I + A^{-1}E) \) and use of previous lemma.

Now we can prove the main theorem:

**THEOREM 1:** Assume that \((A + E)y = b + e_b \) and \( Ax = b \) and that \( \| A^{-1} \| \| E \| < 1 \). Then \( A + E \) is nonsingular and
\[
\frac{\| x - y \|}{\| x \|} \leq \frac{\| A^{-1} \| \| A \|}{1 - \| A^{-1} \| \| E \|} \left( \| E \| + \| e_b \| \right)
\]
Proof: From \((A + E)y = b + e_b\) and \(Ax = b\) we get
\((A + E)(y - x) = e_b - Ex\). Hence:
\[
\begin{align*}
  y - x &= (A + E)^{-1}(e_b - Ex) \\
\end{align*}
\]
Taking norms →
\[
\frac{\|y - x\|}{\|x\|} \leq \|(A + E)^{-1}\| \left[\|e_b\|/\|x\| + \|E\|\right]
\]
Dividing by \(\|x\|\) and using result of lemma
\[
\frac{\|y - x\|}{\|x\|} \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|E\|} \left[\|e_b\|/\|x\| + \|E\|\right]
\]
Result follows by using inequality \(\|A\||\|x\| \geq \|b\|\).... QED

Example: Consider, for a large \(\alpha\), the \(n \times n\) matrix
\[
A = I + \alpha e_1 e_1^T
\]
Inverse of \(A\) is : \(A^{-1} = I - \alpha e_1 e_1^T\) ➤ For the \(\infty\)-norm we have
\[
\|A\|_{\infty} = \|A^{-1}\|_{\infty} = 1 + |\alpha|
\]
so that
\[
\kappa_{\infty}(A) = (1 + |\alpha|)^2.
\]
Can give a very large condition number for a large \(\alpha\) – but all the eigenvalues of \(A\) are equal to one.

The quantity \(\kappa(A) = \|A\| \|A^{-1}\|\) is called the condition number of the linear system with respect to the norm \(\|\cdot\|\). When using the \(p\)-norms we write:
\[
\kappa_p(A) = \|A\|_p \|A^{-1}\|_p
\]

Note: \(\kappa_2(A) = \sigma_{\text{max}}(A)/\sigma_{\text{min}}(A)\) = ratio of largest to smallest singular values of \(A\). Allows to define \(\kappa_2(A)\) when \(A\) is not square.

Determinant *is not* a good indication of sensitivity

Small eigenvalues *do not* always give a good indication of poor conditioning.

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\(\square\)

Show that \(\kappa(I) = 1\);

Show that \(\kappa(A) \geq 1\);

Show that \(\kappa(A) = \kappa(A^{-1})\)

Show that for \(\alpha \neq 0\), we have \(\kappa(\alpha A) = \kappa(A)\)
Simplification when $e_b = 0$:
$$\frac{\|x - y\|}{\|x\|} \leq \frac{\|A^{-1}\| \|E\|}{1 - \|A^{-1}\| \|E\|} \frac{\|x - y\|}{\|x\|} \leq \|A^{-1}\| \|A\| \frac{\|e_b\|}{\|b\|}$$

Slightly less general form: Assume that $\|E\|/\|A\| \leq \delta$ and $\|e_b\|/\|b\| \leq \delta$ and $\delta \kappa(A) < 1$ then
$$\frac{\|x - y\|}{\|x\|} \leq \frac{2 \delta \kappa(A)}{1 - \delta \kappa(A)}$$

Show the above result

Another common form:

**THEOREM 2:** Let $(A + \Delta A)y = b + \Delta b$ and $Ax = b$ where $\|\Delta A\| \leq \epsilon \|E\|$, $\|\Delta b\| \leq \epsilon \|e_b\|$, and assume that $\epsilon \|A^{-1}\| \|E\| < 1$. Then
$$\frac{\|x - y\|}{\|x\|} \leq \frac{\epsilon \|A^{-1}\| \|A\|}{1 - \epsilon \|A^{-1}\| \|E\|} \left(\frac{\|e_b\|}{\|b\|} + \frac{\|E\|}{\|A\|}\right)$$

Results to be seen later are of this type.

**Normwise backward error**

We solve $Ax = b$ and find an approximate solution $y$

*Question:* Find smallest perturbation to apply to $A, b$ so that *exact* solution of perturbed system is $y$

**Normwise backward error in just $A$ or $b$**

Suppose we model entire perturbation in RHS $b$.

Let $r = b - Ay$ be the residual.

Then $y$ satisfies $Ay = b + \Delta b$ with $\Delta b = -r$ exactly.

The relative perturbation to the matrix is $\|r\|/\|b\|$.

Suppose we model entire perturbation in matrix $A$.

Then $y$ satisfies $(A + ry^T/y^Ty)y = b$

The relative perturbation to the matrix is
$$\frac{\|ry^T\|}{\|y^Ty\|} / \|A\|_2 = \frac{\|r\|_2}{\|A\| \|y\|_2}$$
For a given $y$ and given perturbation directions $E, e_b$, we define the Normwise backward error:

\[
\eta_{E,e_b}(y) = \min \left\{ \epsilon \mid (A + \Delta A)y = b + \Delta b; \right. \\
\left. \|\Delta A\| \leq \epsilon \|E\|; \quad \|\Delta b\| \leq \epsilon \|e_b\| \right\}
\]

In other words $\eta_{E,e_b}(y)$ is the smallest $\epsilon$ for which

\[
\begin{align*}
(A + \Delta A)y &= b + \Delta b; \\
\|\Delta A\| &\leq \epsilon \|E\|; \\
\|\Delta b\| &\leq \epsilon \|e_b\|
\end{align*}
\]

- $\|y\|$ is given (a computed solution). $E$ and $e_b$ to be selected (most likely 'directions of perturbation for $A$ and $b'$).

- Typical choice: $E = A, e_b = b$

\[\Delta 6\] Explain why this is not unreasonable

Let $r = b - Ay$. Then we have:

**Theorem 3:**

\[
\eta_{E,e_b}(y) = \frac{\|r\|}{\|E\|\|y\| + \|e_b\|}
\]

Normwise backward error is for case $E = A, e_b = b$:

\[
\eta_{A,b}(y) = \frac{\|r\|}{\|A\|\|y\| + \|b\|}
\]

\[\Delta 7\] Show how this can be used in practice as a means to stop some iterative method which computes a sequence of approximate solutions to $Ax = b$.

\[\Delta 8\] Consider the $6 \times 6$ Vandermonde system $Ax = b$ where $a_{ij} = j^2(i-1)$, $b = A^* [1, 1, \ldots, 1]^T$. We perturb $A$ by $E$, with $|E| \leq 10^{-10}|A|$ and $b$ similarly and solve the system. Evaluate the backward error for this case. Evaluate the forward bound provided by Theorem 2. Comment on the results.
Estimating condition numbers.

- Often we just want to get a lower bound for condition number [it is ‘worse than ...’].
- We want to estimate \( \|A\|\|A^{-1}\| \).
- The norm \( \|A\| \) is usually easy to compute but \( \|A^{-1}\| \) is not.
- We want: Avoid the expense of computing \( A^{-1} \) explicitly.

**Idea:**
- Select a vector \( v \) so that \( \|v\| = 1 \) but \( \|Av\| = \tau \) is small.
- Then: \( \|A^{-1}\| \geq \frac{1}{\tau} \) (show why) and:
  \[
  \kappa(A) \geq \frac{\|A\|}{\tau}
  \]

Condition numbers and near-singularity

- \( 1/\kappa \approx \) relative distance to nearest singular matrix.

Let \( A, B \) be two \( n \times n \) matrices with \( A \) nonsingular and \( B \) singular. Then

\[
\frac{1}{\kappa(A)} \leq \frac{\|A - B\|}{\|A\|}
\]

Proof: \( B \) singular \( \rightarrow \exists x \neq 0 \) such that \( Bx = 0 \).

\[
\|x\| = \|A^{-1}Ax\| \leq \|A^{-1}\| \|Ax\| = \|A^{-1}\| \|(A-B)x\| \leq \|A^{-1}\| \|A - B\| \|x\|
\]

Divide both sides by \( \|x\| \times \kappa(A) = \|x\|\|A\| \|A^{-1}\| \rightarrow \) result.

QED.

Example:

Let \( A = \begin{pmatrix} 1 & 1 \\ 1 & 0.99 \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \)

Then \( \frac{1}{\kappa_1(A)} \leq \frac{0.01}{2} \rightarrow \kappa_1(A) \geq \frac{2}{0.01} = 200. \)

It can be shown that (Kahan)

\[
\frac{1}{\kappa(A)} = \min_B \left\{ \frac{\|A-B\|}{\|A\|} \mid \det(B) = 0 \right\}
\]
Estimating errors from residual norms

Let \( \tilde{x} \) an approximate solution to system \( Ax = b \) (e.g., computed from an iterative process). We can compute the residual norm:

\[
\|r\| = \|b - A\tilde{x}\|
\]

Question: How to estimate the error \( \|x - \tilde{x}\| \) from \( \|r\| \)?

➢ One option is to use the inequality

\[
\frac{\|x - \tilde{x}\|}{\|x\|} \leq \kappa(A) \frac{\|r\|}{\|b\|}.
\]

➢ We must have an estimate of \( \kappa(A) \).

Proof of inequality.

First, note that \( A(x - \tilde{x}) = b - A\tilde{x} = r \). So:

\[
\|x - \tilde{x}\| = \|A^{-1}r\| \leq \|A^{-1}\| \|r\|
\]

Also note that from the relation \( b = Ax \), we get

\[
\|b\| = \|Ax\| \leq \|A\| \|x\| \rightarrow \|x\| \geq \frac{\|b\|}{\|A\|}
\]

Therefore,

\[
\frac{\|x - \tilde{x}\|}{\|x\|} \leq \frac{\|A^{-1}\|}{\|b\| / \|A\|} \|r\| = \kappa(A) \frac{\|r\|}{\|b\|}
\]

Show that

\[
\frac{\|x - \tilde{x}\|}{\|x\|} \geq \frac{1}{\kappa(A)} \|r\| / \|b\|.
\]