DETERMINANTS CHAP. 3

Determinants: summary of main results

➤ A determinant of an $n \times n$ matrix is a real number associated with this matrix. Its definition is complex for the general case $\longrightarrow$ We start with $n = 2$ and list important properties for this case.

- Determinant of a 2×2 matrix is:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

- Notation : $\det(A)$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

➤ Next we list the main properties of determinants. These properties are also true for $n \times n$ case to be defined later.

Det(A) for $n \times n$ case can be defined from GE when permutation is not used: Det(A) = product of pivots in GE. More on this later.

➤ Properties written for columns (easier to write) but are also true for rows

Notation: We let $A = [u, v]$ columns u, and v are in $\mathbb{R}^2$.

1 If $v = \alpha u$ then $\det(A) = 0$.

➤ Determinant of linearly dependent vectors is zero

➤ If any one column is zero then determinant is zero

2 Interchanging columns or rows:

$$\det[v, u] = -\det[u, v]$$

3 Linearity:

$$\det[u, \alpha v + \beta w] = \alpha \det[u, v] + \beta \det[u, w]$$

- $\det(A)$ = linear function of each column (individually)
- $\det(A)$ = linear function of each row (individually)

✍ What is the determinant $\det[u, v + \alpha u]$?

4 Determinant of transpose

$$\det(A) = \det(A^T)$$

5 Determinant of Identity

$$\det(I) = 1$$

6 Determinant of a diagonal:

$$\det(D) = d_1 d_2 \cdots d_n$$

7 Determinant of a triangular matrix (upper or lower)

$$\det(T) = a_{11}a_{22}\cdots a_{nn}$$

8 Determinant of product of matrices [IMPORTANT]

$$\det(AB) = \det(A)\det(B)$$

9 Consequence: Determinant of inverse

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

✍ What is the determinant of αA (for 2×2 matrices)?

✍ What can you say about the determinant of a matrix which satisfies $A^2 = I$?

✍ Is it true that $\det(A + B) = \det(A) + \det(B)$?

Determinants – 3×3 case

➤ We will define $\mathbf{3 \times 3}$ determinants from $\mathbf{2 \times 2}$ determinants:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

➤ This is an expansion of the det. with respect to its 1st row.

1st term $= a_{11} \times$ det of matrix obtained by deleting 1st row and 1st column.

2nd term $= -a_{12} \times$ det of matrix obtained by deleting row 1 and column 2. Note the sign change.

3rd term $= a_{13} \times$ det of matrix obtained by deleting row 1 and column 3.

✍ Calculate $\begin{vmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \\ -1 & 2 & 1 \end{vmatrix}$

➤ We will now generalize this definition to any dimension recursively. Need to define following notation.

We will denote by A_{ij} is the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from the matrix A.

Example: If $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \\ -1 & 2 & 1 \end{bmatrix}$ Then: $A_{11} = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$;

$$A_{12} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \; A_{13} = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}; \; A_{23} = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$

Definition The determinant of a matrix $A = [a_{ij}]$ is the sum

$$\det(A) = + a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - a_{14}\det(A_{14}) + \cdots + (-1)^{1+n} a_{1n}\det(A_{1n})$$

- Note the alternating signs
- We can write this as :

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j}\det(A_{1j})$$

- This is an expansion with respect to the 1st row.

Generalization: Cofactors

Define $\boxed{c_{ij} = (-1)^{i+j} \text{det } A_{ij}}$ = cofactor of entry i, j

➤ Then we get a more general expansion formula:

- det (A) can be expanded with respect to i-th as follows

$$\text{det}\,(A) = a_{i1}c_{i1} + a_{i2}c_{i2} + \cdots + a_{in}c_{in}$$

➤ Note i is fixed. Can be done for any i [same result each time]

➤ Case $i = 1$ corresponds to definition given earlier

➤ Similar expressions for expanding w.r.t. column j (now j is fixed)

$$\text{det}\,(A) = a_{1j}c_{1j} + a_{2j}c_{2j} + \cdots + a_{nj}c_{nj}$$

Computing determinants using cofactors

✍ Compute the following determinant by using co-factors. Expand with respect to 1st row.

$$\begin{vmatrix} \mathbf{-1} & \mathbf{2} & \mathbf{0} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} \\ \mathbf{-1} & \mathbf{0} & \mathbf{2} \end{vmatrix}$$

✍ Compute the above determinant by using co-factors. Expand with respect to last row. Then expand with respect to last column.

✍ Compute the following determinant [expand with respect to last row!]

$$\begin{vmatrix} \mathbf{-1} & \mathbf{2} & \mathbf{0} & \mathbf{1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} & \mathbf{2} \\ \mathbf{-1} & \mathbf{0} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{3} \end{vmatrix}$$

✍ Suppose two rows of $\boldsymbol{A}$ are swapped. Use the above definition to show that the determinant changes signs.

✍ Let B be the matrix obtained from a matrix A by multiplying a certain row (or column) of A by a scalar α. Use the definition to show that: $det(B) = \alpha det(A)$.

✍ What is the computational cost of evaluating the determinant using co-cofactor expansions? [Hint: It is BIG!]

✍ Compute F_2, F_3, F_4 when F_n is the n-th dimensional determinant:

$$F_n = \begin{vmatrix} 1 & -1 & & & \\ 1 & 1 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 1 & -1 \\ & & & 1 & -1 \end{vmatrix}$$

✍ (continuation) Challenge: Show a recurrence relation between F_n, F_{n-1} and F_{n-2}. Do you recognize this relation? Compute the first 8 values of F_n

Inverse of a matrix

➤ Let $C = \{c_{ij}\}$ the matrix of cofactors. Entry (i, j) of C has cofactor c_{ij}. Then it is easy to prove that:

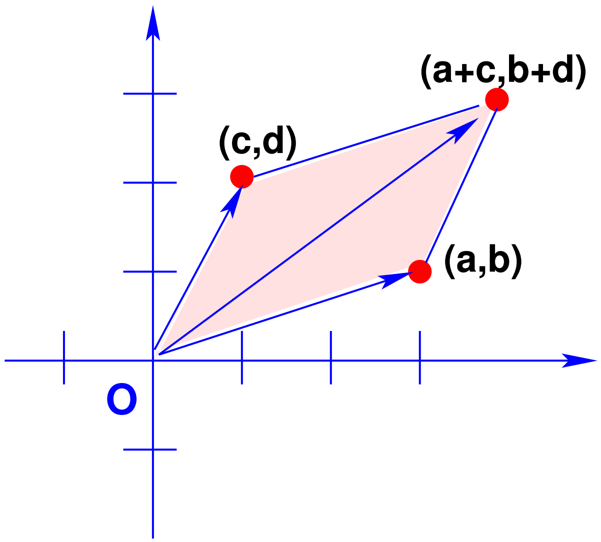
$$A^{-1} = \frac{1}{\det(A)} C^T$$

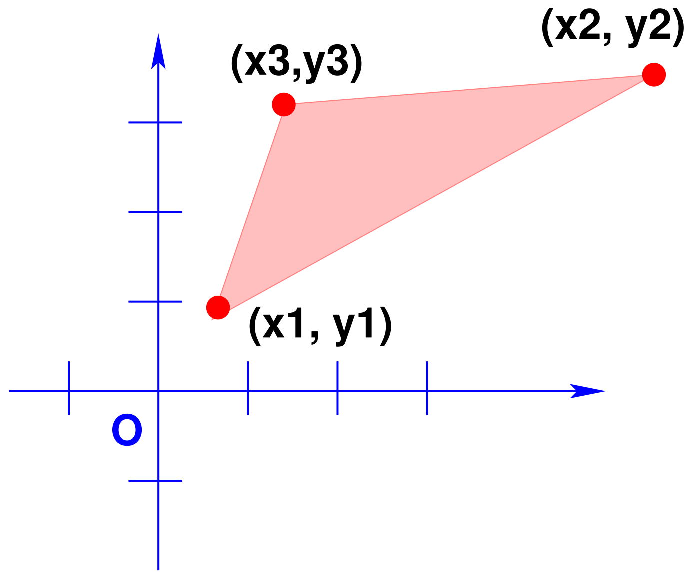
Example: Inverse of a 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

✍ Find the inverses of : $\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$.

Areas in $\mathbb{R}^2$





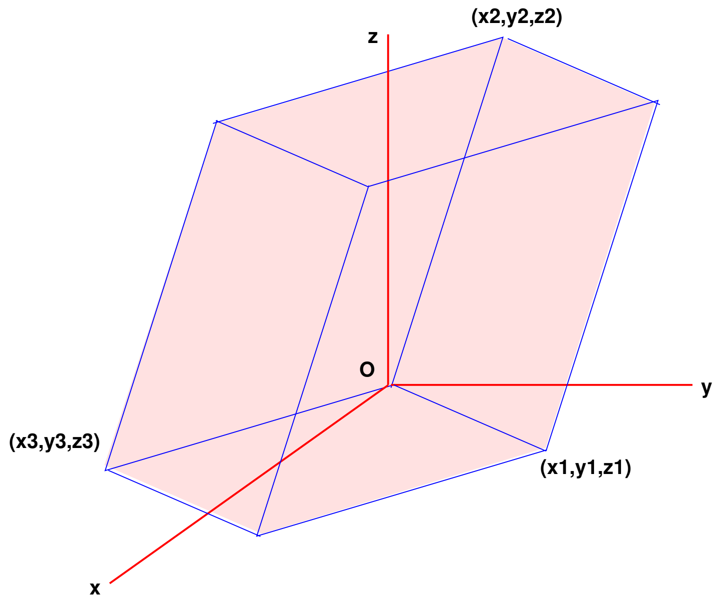
Left figure: Area of a parallelogram spanned by points $(0,0)$, (a,b), (c,d), $(a+c, b+d)$ is:

$$\left| \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right|$$

Right figure: Area of triangle spanned by the points (x_1, y_1), (x_2, y_2), (x_3, y_3) is:

$$\left| \frac{1}{2} \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \right|$$

Volumes in $\mathbb{R}^3$



Volume of parallelepiped spanned by origin and the 3 points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is:

$$\left|\det \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}\right|$$

In summary: Volume $(\mathbb{R}^3)$ /area $(\mathbb{R}^2)$ of a box is $|\det(A)|$ when the box edges are the rows of A.

Areas, Volumes, and Mappings

➤ Determinants are all about areas/volumes – Text has a lot more detail

➤ See section "Determinants as area or volume" in text

✍ See example 4 in same section

➤ Linear mappings and determinants [p. 184 in text]

Q: if a region in $\mathbb{R}^2$ is transformed linearly (i.e., by a linear mapping $\boldsymbol{T}$) – how does its area change?

A: it is multiplied by the | determinant | of the matrix representing $\boldsymbol{T}$. Stated in next theorem

Theorem Let T the linear mapping from/to $\mathbb{R}^2$ represented by a matrix A. If S is a parallelogram in $\mathbb{R}^2$ then

$$\{\text{area of } T(S)\ \} = |\det(A)|\ .\ \{\text{area of } S\ \}$$

Similarly, if $T$is the linear mapping from/to $\mathbb{R}^3$ represented by a matrix A and S is a parallelipiped in $\mathbb{R}^3$ then

$$\{\text{volume of } T(S)\ \} = |\det(A)|\ .\ \{\text{volume of } S\ \}$$

➤ Important point: Results also true for any region in $\mathbb{R}^2$ (1st part) and $\mathbb{R}^3$ (2nd part)

✍ See Example 4 in Section 3.2 which uses this to compute the area of an ellipse.

How to compute determinants in practice?

- Co-factor expansion?? *Not practical*. Instead:
- Perform an LU factorization of A with pivoting.
- If a zero column is encountered LU fails but $det(A) = 0$
- If not get det $=$ product of diagonal entries multiplied by a sign ± 1 depending on how many times we interchanged rows.

Compute the determinants of the matrices

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 5 & 9 \\ 1 & 0 & -12 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 & 1 & 2 \\ 1 & -2 & -1 & 1 \\ 2 & 0 & 2 & 0 \\ -1 & 1 & -1 & -1 \end{bmatrix}$$