LINEAR INDEPENDENCE AND BASES[4.3]

Recall: Linear independence

➤ The set $\{v_1, ..., v_p\}$ is said to be linearly dependent if there exist weights $\alpha_1, ..., \alpha_p$, not all zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_p v_p = 0 \qquad (1)$$

➤ It is linearly independent otherwise

➤ The above equation is called linear dependence relation among the vectors $v_1, \cdots, v_p$

➤ The set $v_1, v_2, \cdots, v_p$ is linearly dependent if and only if equation (1) has a nontrivial solution, i.e., if there are some weights, $\alpha_1, ..., \alpha_p$, not all zero, such that (1) holds.

➤ In such a case, (1) is called a linear dependence relation among $v_1, ..., v_p$.

Theorem: An indexed set $\{v_1, ..., v_p\}$ of two or more vectors, with $v_1 \neq 0$, is linearly dependent if and only if some v_j (with $j > 1$) is a linear combination of the preceding vectors, $v_1, ..., v_{j-1}$.

✍ As an exercise prove formally this theorem

Definition: Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{b_1, ..., b_p\}$ in V is a basis for H if:

1. $\mathcal{B}$ is a linearly independent set, and
2. The subspace spanned by $\mathcal{B}$ coincides with H; that is, $H = \text{span}\{b_1, ..., b_p\}$

- The definition of a basis applies to the case when $H = V$, (any vector space is a subspace of itself)

- A basis of V is a linearly independent set that spans V.

- Note that condition (2) implies that each of the vectors $b_1, ..., b_p$ must belong to H, because $\mathbf{span}\{b_1, ..., b_p\}$ contains $b_1, ..., b_p$.

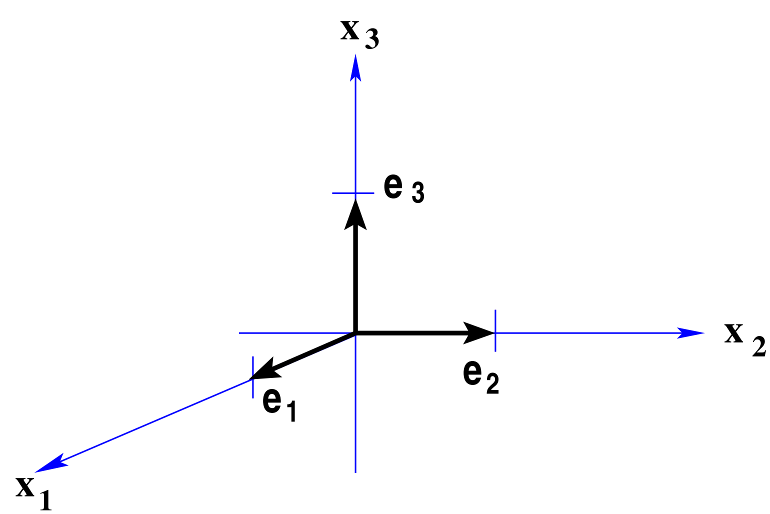
Standard basis of $\mathbb{R}^n$

Let $e_1, ..., e_n$ be the columns of the $n \times n$ matrix, I_n.

That is,

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \; e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}; \; \cdots \; ; e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix};$$

- The set $\{e_1, \cdots, e_n\}$ is called the standard basis for $\mathbb{R}^n$.
- Sometimes the term canonical basis is used



Spanning set theorem

Theorem: Let $S = \{v_1, ..., v_p\}$ be a set in V, and let $H = \text{span}\{v_1, ..., v_p\}$.

1. If one of the vectors in S–say, v_k–is a linear combination of the remaining vectors in S, then the set formed from S by removing v_k still spans H.
2. If $H \neq \{0\}$, some subset of S is a basis for H.

Proof: 1. By rearranging the list of vectors in S, if necessary, we may assume that v_k is the last vector of the list, i.e., v_p, so:

$$v_p = a_1 v_1 + ... + a_{p-1} v_{p-1} \tag{1}$$

➤ Given any x in H, we may write

$$x = \alpha_1 v_1 + ... + \alpha_{p-1} v_{p-1} + \alpha_p v_p \tag{2}$$

for suitable scalars $\alpha_1, ..., \alpha_p$.

➤ Substituting the expression for v_p from (1) into (2) it is easy to see that x is a linear combination of $v_1, ...v_{p-1}$.

➤ Vector x was arbitrary – Thus $\{v_1, ..., v_{p-1}\}$ spans H -

2. If the original spanning set S is linearly independent, then it is already a basis for H.

➤ Otherwise, one of the vectors in S depends on the others and can be deleted, by part (1).

➤ Repeat this process until the spanning set is linearly independent and hence is a basis for H. (If the spanning set is eventually reduced to one vector, that vector will be nonzero because $H \neq \{0\}$) ■

Let $\boldsymbol{H} = \mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ with

$$\boldsymbol{v_1} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}; \quad \boldsymbol{v_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad \boldsymbol{v_3} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix};$$

Show that $\boldsymbol{v_3}$ is a linear combination of the first 2 vectors and then find a basis of $\boldsymbol{H}$.

Basis of Col(A)

Theorem: The pivot columns of a matrix A form a basis for Col(A).

Proof: Let B be the reduced echelon form of A. The set of pivot columns of B is linearly independent (no vector in the set is a linear combination of the vectors that precede it).

$$B = \begin{bmatrix} 1 & * & 0 & * & * & * & 0 & * & * & 0 & * \\ & & 1 & * & * & * & 0 & * & * & 0 & * \\ & & & & & & 1 & * & * & 0 & * \\ & & & & & & & & & 1 & * \\ & & & & & & & & & & \end{bmatrix}$$

- Since A is row equivalent to B, pivot columns of A are lin. independent too
- Every nonpivot column of A is a linear combination of the pivot columns of A.

- Thus the nonpivot columns of a may be discarded from the spanning set for Col($\boldsymbol{A}$), by the Spanning Set Theorem.
- This leaves the pivot columns of $\boldsymbol{A}$ as a basis for Col($\boldsymbol{A}$). ■

Note: The pivot columns of a matrix $\boldsymbol{A}$ are evident when $\boldsymbol{A}$ has been reduced to an echelon form $\boldsymbol{B}$ (standard or reduced). However be sure to use the pivot columns of $\boldsymbol{A}$ itself for the basis of Col($\boldsymbol{A}$), not those of $\boldsymbol{B}$

Two Views of a Basis:

➤ In the Spanning Set Theorem, the deletion of vectors from a spanning set stops when the set becomes linearly independent.

➤ If one more vector is deleted, this vector is not a linear combination of the remaining vectors $\longrightarrow$ the smaller set will no longer span $\boldsymbol{V}$

➤ Thus a basis is a spanning set that is as small as possible.

➤ A basis is also a linearly independent set that is as large as possible.

➤ If $\boldsymbol{S}$ is a basis for $\boldsymbol{V}$, and if $\boldsymbol{S}$ is enlarged by one vector –say, $\boldsymbol{w}$–from $\boldsymbol{V}$, then the new set loses linear independence [Explain why]

Dimension and rank

➤ It can be shown that the number of vectors in a basis of a subspace H is always the same –

✍ See Theorems 9 and 10 in sect. 4.5 of *text* for details

✍ Take $U = [u_1, u_2, u_3]$ and a basis $B = [v_1, v_2]$ of the space. Show that there is a matrix $G \in \mathbb{R}^{2\times 3}$ such that $U = VG$. Show that there is a vector w such that $Gw = 0$. Conclude that the columns of U must be dependent.

➤ Hence the definition:

Definition: The **dimension** of a subspace H is the number of vectors in any basis for H. Special case: If $H = \{0\}$, $\dim(H)$ is zero.

➤ Notation $\dim(H)$

- Related (and important) definition

Definition: The rank of a matrix A is the dimension of its column space.

- Notation: $\text{rank}(A)$.

- Note: $\text{rank}(A) =$ number of pivot columns in A.

- Recall from an earlier example that we could find a spanning set of $\text{Nul}(A)$ which has as many vectors as there are free variables.

- Therefore $\mathbf{dim}(\text{Nul}(A)) =$ number of free variables. Hence the important result

$$\text{rank}(A) + \mathbf{dim}(\text{Nul}(A)) = n$$

- Known as the Rank+Nullity theorem

- $\text{rank}(A) = \text{rank}(A^T)$ [row-rank=column rank]