INTERPOLATE WITH DIVIDED DIFFERENCES  DEMO

Goal: interpolate at T=[-2 0 1 -1] Y=[-27 -1 0 8]
Update from: interpolate at T=[-2 0 1] Y=[-27 -1 0]

\[ p_{2a} = -4t^2 + 5t - 1 \]

correction polynomial (zero at the old knots)

\[ w_{3a} = (t+2)(t-1) = 1t^3 + 1t^2 - 2t + 0 \]

Evaluate polynomials at the new knot T=-1:

\[ p_{2a}(-1) = -10, w_{3a}(-1) = 2 \]

to get amount of correction:

\[ \alpha_3 = (Y_4 - p_{2a}(-1))/w_{3a}(-1) = 9 \]

Result: polynomial interpolating at all four points T=[-2;0;1;-1] Y=[-27;-1;0;8]

\[ p_3 = p_{2a} + \alpha_3 w_{3a} = 9t^3 + 5t^2 - 13t - 1 \]

Verify the new polynomial actually goes thru all four points

\[ p_3(T_k) = [-27 -1 0 8] \]

Now show similar procedure, but starting with last 3 points instead:

from: interpolate at T=[0 1 -1] Y=[-1 0 8]

\[ p_{2b} = 5t^2 - 4t - 1 \]

correction polynomial (zero at the old knots)

\[ w_{3b} = (t)(t-1)(t+1) = 1t^3 + 0t^2 - t + 0 \]

\[ \alpha_3 = (Y_1 - p_{2b}(-2))/w_{3b}(-2) = 9 \text{ (same as } \alpha_3 \text{ found above)} \]

\[ p_3 = p_{2b} + \alpha_3 w_{3b} = 9t^3 + 5t^2 - 13t - 1 \]

Divided Difference: start with polynomial interpolating T=[0 1] Y=[-1 0 0]:

\[ p_1 = 1t - 1 \]

\[ w_2 = (t)(t-1) = 1t^2 - t + 0 \]

\[ \alpha_{2a} = -4 \text{ (obtained by adding new point } T_1 = -2) \]

\[ p_{2a} = p_1 + \alpha_{2a} w_2 = -4t^2 + 5t - 1 \]

\[ \alpha_{26} = 5 \text{ (obtained by adding new point } T_4 = -1) \]

\[ p_{26} = p_1 + \alpha_{26} w_2 = 5t^2 - 4t - 1 \]

So, equate the two ways to express the next polynomial:

\[ p_3 = p_{2a} + \alpha_3 w_{3a} = p_{2b} + \alpha_3 w_{3b} \]

subtract \( p_1 \) and divide by the common factor \( w_2 \):

\[ \alpha_{2a} + \alpha_3(t-T_1) = \alpha_{26} + \alpha_3(t-T_4) \]

Solve for Divided Difference:

\[ p_{26} - p_{2a} = 9t^2 - 9t = 9(t)(t-1) \]

\[ = (\alpha_{2b} - \alpha_{2a})w_2 \text{ (agree at shared knots 0,1)} \]

\[ w_{3b} - w_{3a} = t(t-1)(t+1) - (t+2)(t-1) \]

\[ = [t(t-1) \cdot [(t+1) - (t+2)] = t(t-1) \cdot [-1] \]

\[ 0 = [p_{2b} + \alpha_3 w_{3b}] - [p_{2a} + \alpha_3 w_{3a}] \]

\[ = [t(t-1) \cdot [9 + \alpha_3(-1)] \]

\[ = w_2 \cdot [(\alpha_{2b} - \alpha_{2a}) - \alpha_3(T_4 - T_1)] \]

\[ \alpha_3 = (\alpha_{2a} - \alpha_{2b})/(T_1 - T_4) = 9 \]

All divided differences

\[
\begin{array}{c|c|c}
T & Y & \text{Div Diffs = } f[t_1, \ldots, t_k] \\
\hline
-2 & -27 & 27 \\
-1 & -1 & 9 \\
0 & 0 & 5 \\
1 & -1 & 4 \\
-1 & 8 & 8 \\
\end{array}
\]

So the final polynomial can be written in different ways, e.g.:

\[ p_3 = -27 + 13(t+2) - 4(t+2)(t) + 9(t+2)(t-1) \]

\[ p_3 = 8 - 4(t+1) + 5(t+1)(t-1) + 9(t+1)(t-1)(t) \]

\[ p_3 = 0 + 1(t-1) + 5(t-1)(t-1) + 9(t-1)(t+1) \]

\[ p_3 = 0 + 1(t-1) - 4(t-1)(t) + 9(t-1)(t+2) \]
error bound

Suppose \( p(t) \) interpolates \( f(t) \) at the \( n+1 \) points \( t_0, \ldots, t_n \). Choose an arbitrary new point \( \hat{t} \) such that \( t_0 \leq \hat{t} \leq t_n \). We wish to find an error estimate for \( f(\hat{t}) - p(\hat{t}) \). For this purpose, define

\[
e(\hat{t}) = f(\hat{t}) - [p(\hat{t}) + \hat{\alpha} w(\hat{t})]
\]

where \( w(t) = (t-t_0) \cdots (t-t_n) \) and \( \hat{\alpha} \) is chosen so that

\[
0 = f(\hat{t}) - [p(\hat{t}) + \hat{\alpha} w(\hat{t})]
\]  
(1)

Differentiate \( e(\hat{t}) \) \( n+1 \) times (observing that \( p^{(n+1)}(t) = 0 \)) to get

\[
e^{(n+1)}(\hat{t}) = f^{(n+1)}(\hat{t}) - [p^{(n+1)}(\hat{t}) + \hat{\alpha} w^{(n+1)}(\hat{t})]
\]

\[
= f^{(n+1)}(\hat{t}) - \hat{\alpha} (n+1)!
\]  
(2)

The fn \( e(\hat{t}) \) has \( n+2 \) zeros, namely the original knots \( t_0, \ldots, t_n \) and the additional test point \( \hat{t} \). Hence the mean value theorem implies \( e'(\hat{t}) \) has \( n+1 \) zeros which interlace those of \( e(\hat{t}) \). Continuing this argument, we find that \( e^{(n+1)}(\hat{t}) \) must have at least one zero at some point \( \tau \in [t_0, t_n] \). Hence (2) yields

\[
\hat{\alpha} = f^{(n+1)}(\tau)/(n+1)!
\].

The resulting error at the [arbitrary] point \( \hat{t} \) is (from equ. (1))

\[
f(\hat{t}) - p(\hat{t}) = \hat{\alpha} w(\hat{t}) = f^{(n+1)}(\tau) \cdot w(\hat{t})/(n+1)!
\]  
(3)

Compare this to the remainder term in the Taylor expansion: \( f^{(n+1)}(\tau) \cdot (\hat{t} - t_0)^{n+1}/(n+1)! \).

If the knots are equally spaced with stepsize \( h \) so that \( t_k = t_0 + kh \) for \( k = 0, 1, \ldots, n \), then one has the bound in the internal \( [t_0, t_n] \):

\[
|w(t)| = |(t-t_0) \cdots (t-t_n)| \leq h^{n+1} \max_{0<u<1} |(u)(u-1)|n! = \frac{h^{n+1}n!}{4}
\]  
(4)

so that the interpolation error for equally spaced points is bounded by

\[
|e(t)| \leq \max_z |f^{(n+1)}(z)| \frac{h^{n+1}}{4(n+1)}
\]  
for \( t_0 \leq t \leq t_n \).

To derive (4), observe that within the interval \( [t_0, t_n] \), the fn \( w(t) \) is largest in the first and last subintervals \( [t_0, t_1] \) and \( [t_{n-1}, t_n] \). So let \( t \) be a number s.t. \( t_0 \leq t \leq t_1 \). Then

\[
|w(t)| = |(t-t_0)(t-t_1)| \cdots |(t-t_2) \cdots (t-t_n)|
\]

\[
\leq |(t-t_0)(t-t_1)| \cdot (2h)(3h) \cdots (nh)
\]

\[
\leq h^2/4 \cdot (2h)(3h) \cdots (nh)
\]

\[
= h^{n+1} \cdot n!/4
\]  
for \( t_0 \leq t \leq t_1 \)