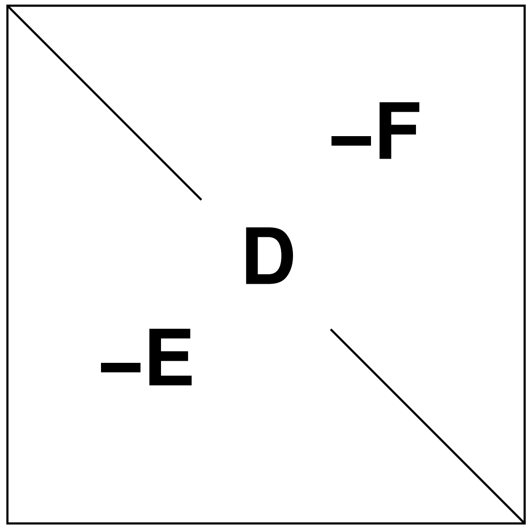
# Basic relaxation techniques

- Relaxation methods: Jacobi, Gauss-Seidel, SOR
- Basic convergence results
- Optimal relaxation parameter for SOR
- See Chapter 4 of text for details.

## *Basic relaxation schemes*

➤ *Relaxation schemes:* methods that modify one component of current approximation at a time

➤ Based on the decomposition $A = D - E - F$ with: $D$ = diag(A), $-E$ = strict lower part of $A$ and $-F$ = its strict upper part.




Gauss-Seidel iteration for solving $Ax = b$:

➤ corrects $j$-th component of current approximate solution, to zero the $j - th$ component of residual for $j = 1, 2, \cdots, n$.

- Gauss-Seidel iteration can be expressed as:

$$(D - E)x^{(k+1)} = Fx^{(k)} + b$$

Can also define a backward Gauss-Seidel Iteration:

$$(D - F)x^{(k+1)} = Ex^{(k)} + b$$

and a Symmetric Gauss-Seidel Iteration: forward sweep followed by backward sweep.

Over-relaxation is based on the splitting:

$$\omega A = (D - \omega E) - (\omega F + (1 - \omega)D)$$

$\longrightarrow$ successive overrelaxation, (SOR):

$$(D - \omega E)x^{(k+1)} = [\omega F + (1 - \omega)D]x^{(k)} + \omega b$$

## Iteration matrices

- Previous methods based on a splitting of $A$: $A = M - N \rightarrow$

$$Mx = Nx + b \quad \rightarrow \quad Mx^{(k+1)} = Nx^{(k)} + b \rightarrow$$

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b \equiv Gx^{(k)} + f$$

Jacobi, Gauss-Seidel, SOR, & SSOR iterations are of the form

$$\begin{aligned} G_{Jac} &= D^{-1}(E + F) = I - D^{-1}A \\ G_{GS} &= (D - E)^{-1}F = I - (D - E)^{-1}A \\ G_{SOR} &= (D - \omega E)^{-1}(\omega F + (1 - \omega)D) \\ &= I - (\omega^{-1}D - E)^{-1}A \\ G_{SSOR} &= I - \omega(2 - \omega)(D - \omega F)^{-1}D(D - \omega E)^{-1}A \end{aligned}$$

## *General convergence result*

Consider the iteration: $x^{(k+1)} = Gx^{(k)} + f$

(1) Assume that $\rho(G) < 1$. Then $I - G$ is non-singular and $G$ has a fixed point. Iteration converges to a fixed point for any $f$ and $x^{(0)}$.

(2) If iteration converges for any $f$ and $x^{(0)}$ then $\rho(G) < 1$.

**Example:** Richardson's iteration

$$x^{(k+1)} = x^{(k)} + \alpha(b - Ax^{(k)})$$

✍ Assume $\Lambda(A) \subset \mathbb{R}$. When does the iteration converge?

## A few well-known results

➤ Jacobi and Gauss-Seidel converge for diagonal dominant matrices, i.e., matrices such that

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, i = 1, \cdots, n$$

➤ SOR converges for $0 < \omega < 2$ for SPD matrices

➤ The optimal $\omega$ is known in theory for an important class of matrices called 2-cyclic matrices or matrices with property A.

- A matrix has property $A$ if it can be (symmetrically) permuted into a $2 \times 2$ block matrix whose diagonal blocks are diagonal.

$$PAP^T = \begin{bmatrix} D_1 & E \\ E^T & D_2 \end{bmatrix}$$

- Let $A$ be a matrix which has property $A$. Then the eigenvalues $\lambda$ of the SOR iteration matrix and the eigenvalues $\mu$ of the Jacobi iteration matrix are related by

$$(\lambda + \omega - 1)^2 = \lambda \omega^2 \mu^2$$

- The optimal $\omega$ for matrices with property $A$ is given by

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho(B)^2}}$$

where $B$ is the Jacobi iteration matrix.

## An observation — Introduction to Preconditioning

➤ The iteration $x^{(k+1)} = Gx^{(k)} + f$ is attempting to solve $(I - G)x = f$. Since $G$ is of the form $G = M^{-1}[M - A]$ and $f = M^{-1}b$, this system becomes

$$M^{-1}Ax = M^{-1}b$$

where for SSOR, for example, we have

$$M_{SSOR} = (D - \omega E)D^{-1}(D - \omega F)$$

referred to as the SSOR 'preconditioning' matrix.

In other words:

*Relaxation iter.* $\Longleftrightarrow$ *Preconditioned Fixed Point Iter.*

# Projection methods

- Introduction to projection-type techniques
- Sample one-dimensional Projection methods
- Some theory and interpretation –
- See Chapter 5 of text for details.

## *Projection Methods*

➤ The main idea of projection methods is to extract an approximate solution from a subspace.

➤ We define a subspace of approximants of dimension $m$ and a set of $m$ conditions to extract the solution

➤ These conditions are typically expressed by orthogonality constraints.

➤ This defines one basic step which is repeated until convergence (alternatively the dimension of the subspace is increased until convergence).

***Example:*** Each relaxation step in Gauss-Seidel can be viewed as a projection step

## *Background on projectors*

➤ A projector is a linear operator that is idempotent:

$$P^2 = P$$

**A few properties:**

- $P$ is a projector iff $I - P$ is a projector
- $x \in \mathrm{Ran}(P)$ iff $x = Px$ iff $x \in \mathrm{Null}(I - P)$
- This means that : $\mathrm{Ran}(P) = \mathrm{Null}(I - P)$ .
- Any $x \in \mathbb{R}^n$ can be written (uniquely) as $x = x_1 + x_2$, $x_1 = Px \in \mathrm{Ran}(P)$ $x_2 = (I - P)x \in \mathrm{Null}(P)$ - So:

$$\mathbb{R}^n = \mathrm{Ran}(P) \oplus \mathrm{Null}(P)$$

✍ Prove the above properties

## *Background on projectors (Continued)*

➤ The decomposition $\mathbb{R}^n = K \oplus S$ defines a (unique) projector $P$:

- From $x = x_1 + x_2$, set $Px = x_1$.
- For this $P$: $\mathrm{Ran}(P) = K$ and $\mathrm{Null}(P) = S$.
- Note: $dim(K) = m$, $dim(S) = n - m$.

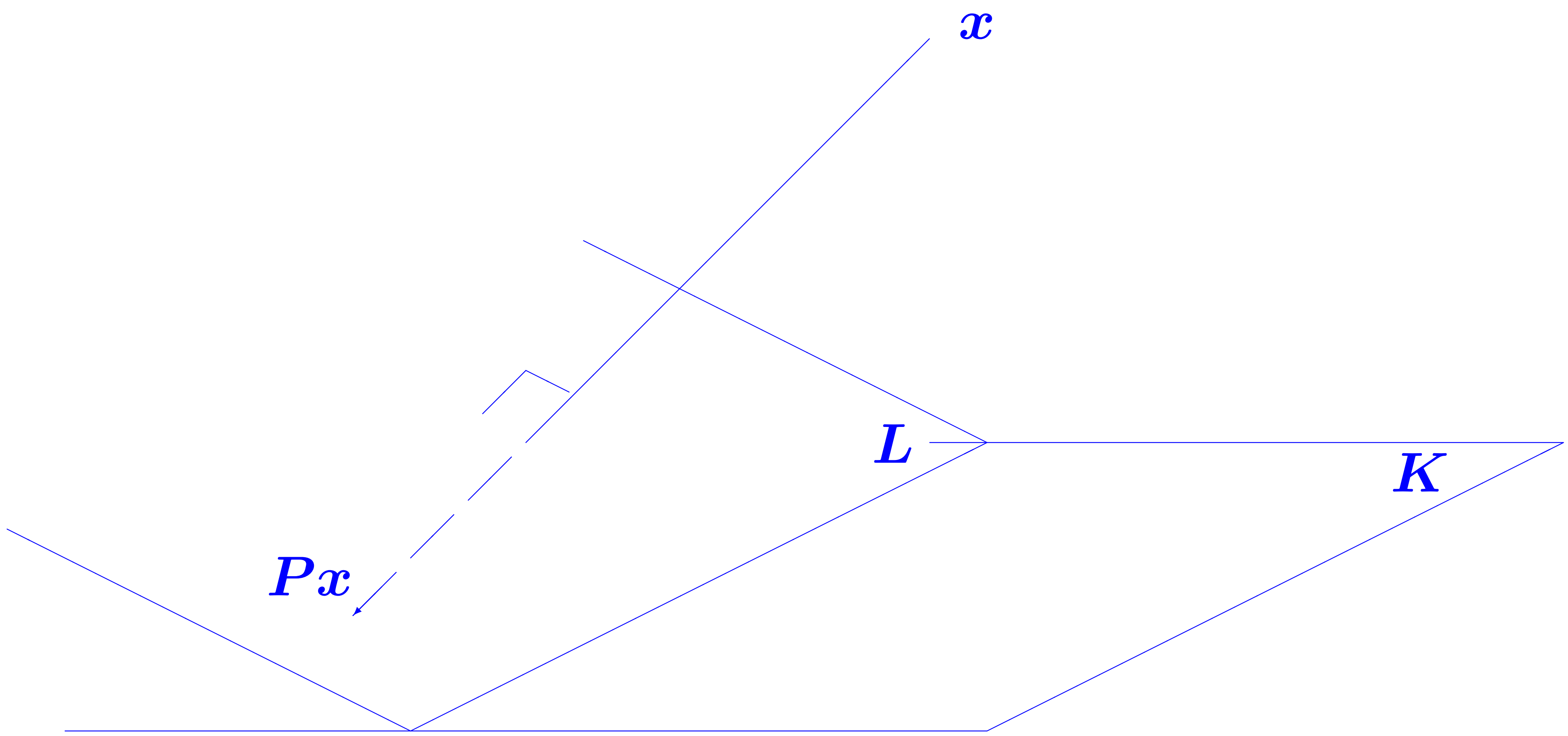
➤ Pb: express mapping $x \to u = Px$ in terms of $K, S$

➤ Note $u \in K$, $x - u \in S$

➤ Express 2nd part with $m$ constraints: let $L = S^\perp$, then

$$u = Px \text{ iff } \begin{cases} u \in K \\ x - u \perp L \end{cases}$$

➤ Projection onto $K$ and orthogonally to $L$




- Illustration: $P$ projects onto $K$ and orthogonally to $L$
- When $L = K$ projector is orthogonal.
- Note: $Px = 0$ iff $x \perp L$.

## *Projection methods*

- Initial Problem: $b - Ax = 0$

Given two subspaces $K$ and $L$ of $\mathbb{R}^N$ define the approximate problem:

$$\text{Find } \tilde{x} \in K \text{ such that } b - A\tilde{x} \perp L$$

- Petrov-Galerkin condition
- $m$ degrees of freedom $(K)$ + $m$ constraints $(L)$ $\rightarrow$
- a small linear system ('projected problem')
- This is a basic projection step. Typically a sequence of such steps are applied

➤ With a nonzero initial guess $x_0$, approximate problem is

$$\text{Find} \quad \tilde{x} \in x_0 + K \quad \text{such that} \quad b - A\tilde{x} \perp L$$

Write $\tilde{x} = x_0 + \delta$ and $r_0 = b - Ax_0$. $\rightarrow$ system for $\delta$:

$$\text{Find } \delta \ \in K \text{ such that } r_0 - A\delta \perp L$$

✍ Formulate Gauss-Seidel as a projection method -

✍ Generalize Gauss-Seidel by defining subspaces consisting of ‘blocks’ of coordinates $\text{span}\{e_i, e_{i+1}, ..., e_{i+p}\}$

## *Matrix representation:*

Let

- $V = [v_1, \ldots, v_m]$ a basis of $K$ &
- $W = [w_1, \ldots, w_m]$ a basis of $L$

➤ Write approximate solution as $\tilde{x} = x_0 + \delta \equiv x_0 + Vy$ where $y \in \mathbb{R}^m$. Then Petrov-Galerkin condition yields:

$$W^T(r_0 - AVy) = 0$$

➤ Therefore,

$$\tilde{x} = x_0 + V[W^TAV]^{-1}W^Tr_0$$

Remark: In practice $W^TAV$ is known from algorithm and has a simple structure [tridiagonal, Hessenberg,..]

## *Prototype Projection Method*

**Until Convergence Do:**

1. Select a pair of subspaces $K$, and $L$;

2. Choose bases: $V = [v_1, \ldots, v_m]$ for $K$ and $W = [w_1, \ldots, w_m]$ for $L$.

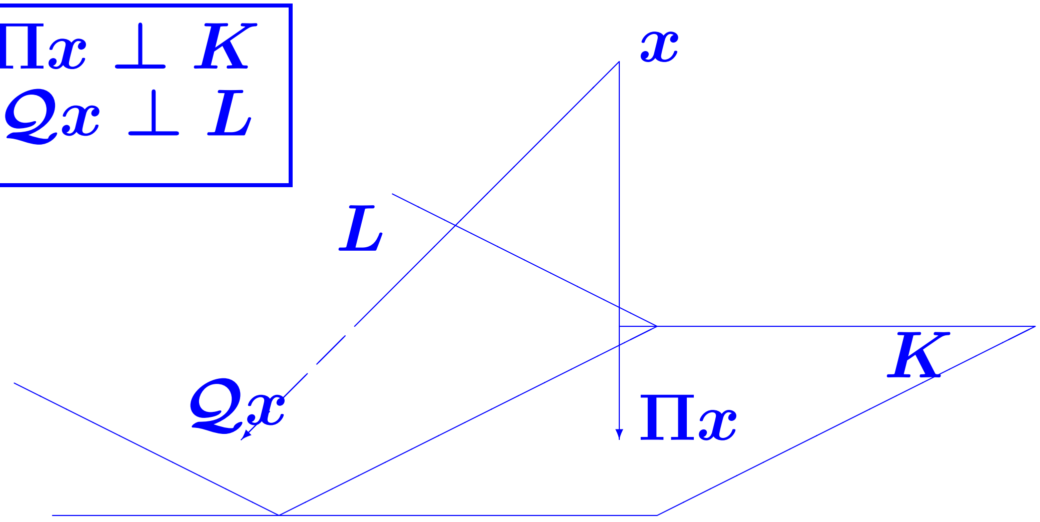
3. Compute :

$$r \leftarrow b - Ax,$$
$$y \leftarrow (W^T AV)^{-1} W^T r,$$
$$x \leftarrow x + Vy.$$

# *Projection methods: Operator form representation*

➤ Let $\mathbf{\Pi}$ = the orthogonal projector onto $\boldsymbol{K}$ and $\boldsymbol{\mathcal{Q}}$ the (oblique) projector onto $\boldsymbol{K}$ and orthogonally to $\boldsymbol{L}$.

$$\boxed{\begin{array}{lll} \mathbf{\Pi}\boldsymbol{x} & \in & \boldsymbol{K}, \quad \boldsymbol{x} - \mathbf{\Pi}\boldsymbol{x} \perp \boldsymbol{K} \\ \boldsymbol{\mathcal{Q}}\boldsymbol{x} & \in & \boldsymbol{K}, \quad \boldsymbol{x} - \boldsymbol{\mathcal{Q}}\boldsymbol{x} \perp \boldsymbol{L} \end{array}}$$




$\mathbf{\Pi}$ and $\boldsymbol{\mathcal{Q}}$ projectors

Assumption: no vector of $\boldsymbol{K}$ is $\perp$ to $\boldsymbol{L}$

In the case $x_0 = 0$, approximate problem amounts to solving

$$\mathcal{Q}(b - Ax) = 0, \quad x \quad \in K$$

or in operator form (solution is $\Pi x$)

$$\mathcal{Q}(b - A\Pi x) = 0$$

**Question:** what accuracy can one expect?

➤ Let $x^*$ be the exact solution. Then

1) We cannot get better accuracy than $\|(I - \Pi)x^*\|_2$, i.e.,

$$\boxed{\|\tilde{x} - x^*\|_2 \geq \|(I - \Pi)x^*\|_2}$$

2) The residual of the exact solution for the approximate problem satisfies:

$$\boxed{\|b - \mathcal{Q}A\Pi x^*\|_2 \leq \|\mathcal{Q}A(I - \Pi)\|_2 \|(I - \Pi)x^*\|_2}$$

## *Two Important Particular Cases.*

**1.** $L = K$

- When $A$ is SPD then $\|x^* - \tilde{x}\|_A = \min_{z \in K} \|x^* - z\|_A$.
- Class of Galerkin or Orthogonal projection methods
- Important member of this class: Conjugate Gradient (CG) method

**2.** $L = AK$.

In this case $\|b - A\tilde{x}\|_2 = \min_{z \in K} \|b - Az\|_2$

- Class of Minimal Residual Methods: CR, GCR, ORTHOMIN, GMRES, CGNR, ...

# *One-dimensional projection processes*

$$K = span\{d\}$$
and
$$L = span\{e\}$$

Then $\tilde{x} = x + \alpha d$. Condition $r - A\delta \perp e$ yields

$$\alpha = \frac{(r,e)}{(Ad,e)}$$

- Three popular choices:

(1) Steepest descent

(2) Minimal residual iteration

(3) Residual norm steepest descent

## 1. Steepest descent.

A is SPD. Take at each step $d = r$ and $e = r$.

Iteration: $$\begin{array}{l} r \leftarrow b - Ax, \\ \alpha \leftarrow (r, r)/(Ar, r) \\ x \leftarrow x + \alpha r \end{array}$$

➤ Each step minimizes $f(x) = \|x - x^*\|_A^2 = (A(x - x^*), (x - x^*))$ in direction $-\nabla f$.

➤ Convergence guaranteed if $A$ is SPD.

✍ As is formulated, the above algorithm requires 2 'matvecs' per step. Reformulate it so only one is needed.

*Convergence* based on the Kantorovitch inequality: Let $B$ be an SPD matrix, $\lambda_{max}$, $\lambda_{min}$ its largest and smallest eigenvalues. Then,

$$\frac{(Bx, x)(B^{-1}x, x)}{(x, x)^2} \leq \frac{(\lambda_{max} + \lambda_{min})^2}{4\,\lambda_{max}\lambda_{min}}, \quad \forall x \neq 0.$$

- This helps establish the convergence result

Let $A$ an SPD matrix. Then, the $A$-norms of the error vectors $d_k = x_* - x_k$ generated by steepest descent satisfy:

$$\|d_{k+1}\|_A \leq \frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} \|d_k\|_A$$

- Algorithm converges for any initial guess $x_0$.

*Proof:* Observe $\|d_{k+1}\|_A^2 = (Ad_{k+1}, d_{k+1}) = (r_{k+1}, d_{k+1})$

➤ by substitution,

$$\|d_{k+1}\|_A^2 = (r_{k+1}, d_k - \alpha_k r_k)$$

➤ By construction $r_{k+1} \perp r_k$ so we get $\|d_{k+1}\|_A^2 = (r_{k+1}, d_k)$. Now:

$$\begin{aligned} \|d_{k+1}\|_A^2 &= (r_k - \alpha_k A r_k, d_k) \\ &= (r_k, A^{-1} r_k) - \alpha_k (r_k, r_k) \\ &= \|d_k\|_A^2 \left(1 - \frac{(r_k, r_k)}{(r_k, A r_k)} \times \frac{(r_k, r_k)}{(r_k, A^{-1} r_k)}\right). \end{aligned}$$

Result follows by applying the Kantorovich inequality. ■

## 2. Minimal residual iteration.

A positive definite ($A + A^T$ is SPD). Take at each step $d = r$ and $e = Ar$.

Iteration:
$$r \leftarrow b - Ax, \\ \alpha \leftarrow (Ar, r)/(Ar, Ar) \\ x \leftarrow x + \alpha r$$

- Each step minimizes $f(x) = \|b - Ax\|_2^2$ in direction $r$.
- Converges under the condition that $A + A^T$ is SPD.

✍ As is formulated, the above algorithm would require 2 'matvecs' at each step. Reformulate it so that only one matvec is required

## Convergence

Let $A$ be a real positive definite matrix, and let

$$\mu = \lambda_{min}(A + A^T)/2, \quad \sigma = \|A\|_2.$$

Then the residual vectors generated by the Min. Res. Algorithm satisfy:

$$\|r_{k+1}\|_2 \leq \left(1 - \frac{\mu^2}{\sigma^2}\right)^{1/2} \|r_k\|_2$$

- In this case Min. Res. converges for any initial guess $x_0$.

*Proof:* Similar to steepest descent. Start with

$$\begin{aligned}\|r_{k+1}\|_2^2 &= (r_k - \alpha_k A r_k, r_k - \alpha_k A r_k)\\ &= (r_k - \alpha_k A r_k, r_k) - \alpha_k (r_k - \alpha_k A r_k, A r_k).\end{aligned}$$

By construction, $r_{k+1} = r_k - \alpha_k A r_k$ is $\perp$ $A r_k$. ➤ $\|r_{k+1}\|_2^2 = (r_k - \alpha_k A r_k, r_k)$. Then:

$$\begin{aligned}\|r_{k+1}\|_2^2 &= (r_k - \alpha_k A r_k, r_k)\\ &= (r_k, r_k) - \alpha_k (A r_k, r_k)\\ &= \|r_k\|_2^2 \left(1 - \frac{(A r_k, r_k)}{(r_k, r_k)} \frac{(A r_k, r_k)}{(A r_k, A r_k)}\right)\\ &= \|r_k\|_2^2 \left(1 - \frac{(A r_k, r_k)^2}{(r_k, r_k)^2} \frac{\|r_k\|_2^2}{\|A r_k\|_2^2}\right).\end{aligned}$$

Result follows from the inequalities $(Ax, x)/(x, x) \geq \mu > 0$ and $\|A r_k\|_2 \leq \|A\|_2 \, \|r_k\|_2$. ■

## 3. Residual norm steepest descent.

A is arbitrary (nonsingular). Take at each step $d = A^T r$ and $e = Ad$.

Iteration: $$\begin{array}{l} r \leftarrow b - Ax, d = A^T r \\ \alpha \leftarrow \|d\|_2^2 / \|Ad\|_2^2 \\ x \leftarrow x + \alpha d \end{array}$$

- Each step minimizes $f(x) = \|b - Ax\|_2^2$ in direction $-\nabla f$.
- Important Note: equivalent to usual steepest descent applied to normal equations $A^T Ax = A^T b$ .
- Converges under the condition that $A$ is nonsingular.